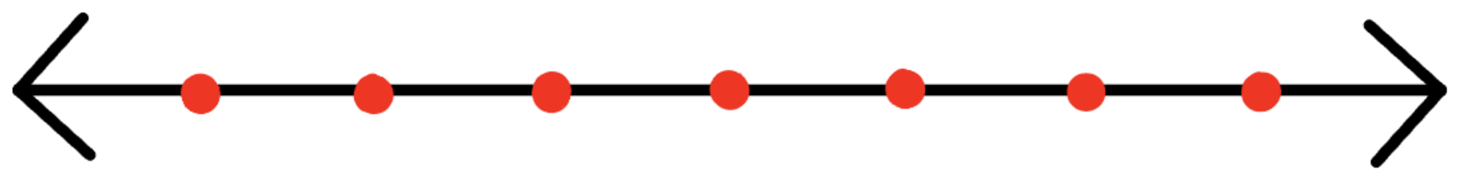
The asymptotic spectrum distance and the Shannon capacity

Pjotr Buys

Question

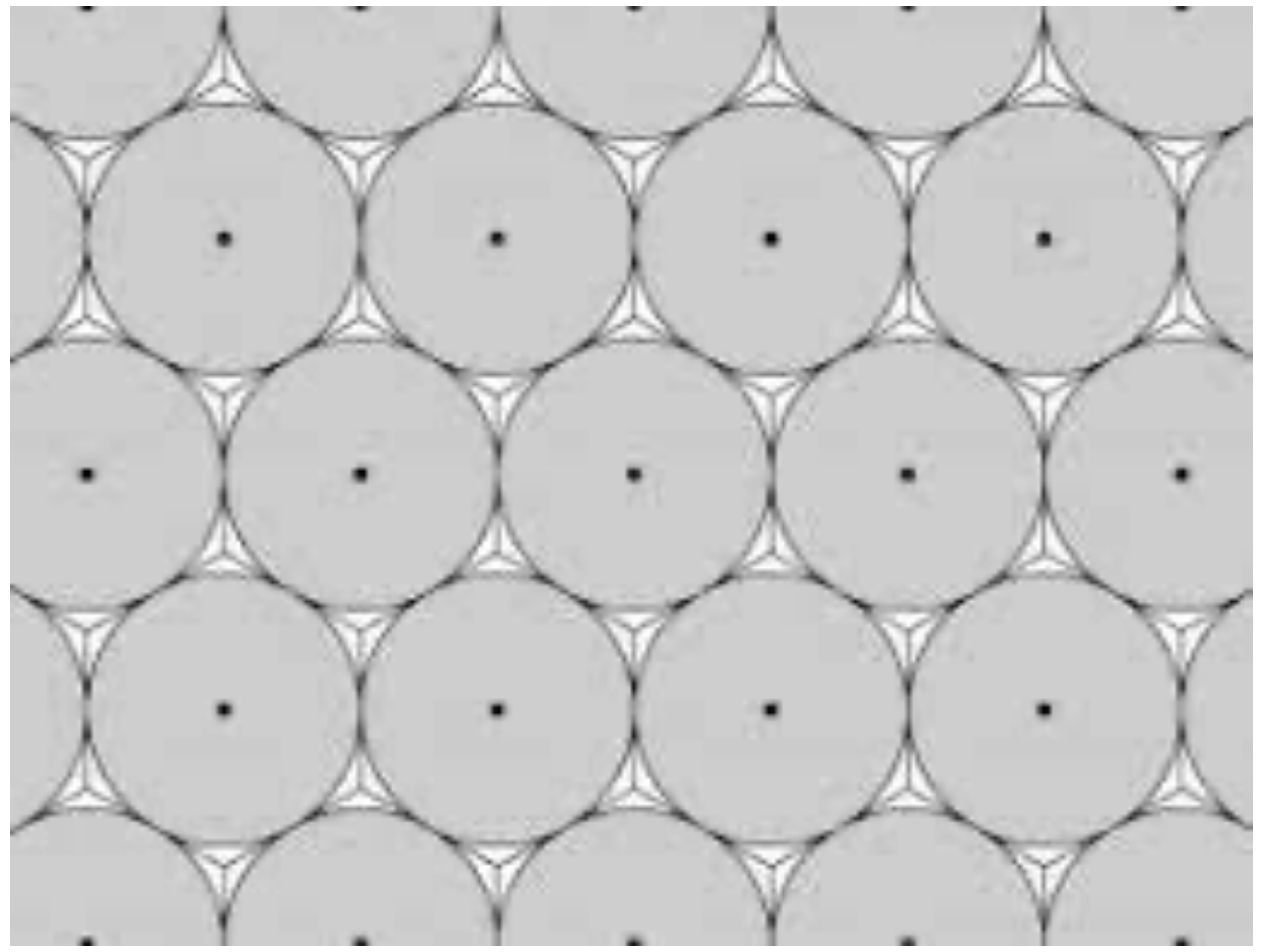
What is the maximal density of $S \subseteq \mathbb{R}^n$ satisfying $|x - y| \geq 1$ for distinct $x, y \in S$.

$n = 1$



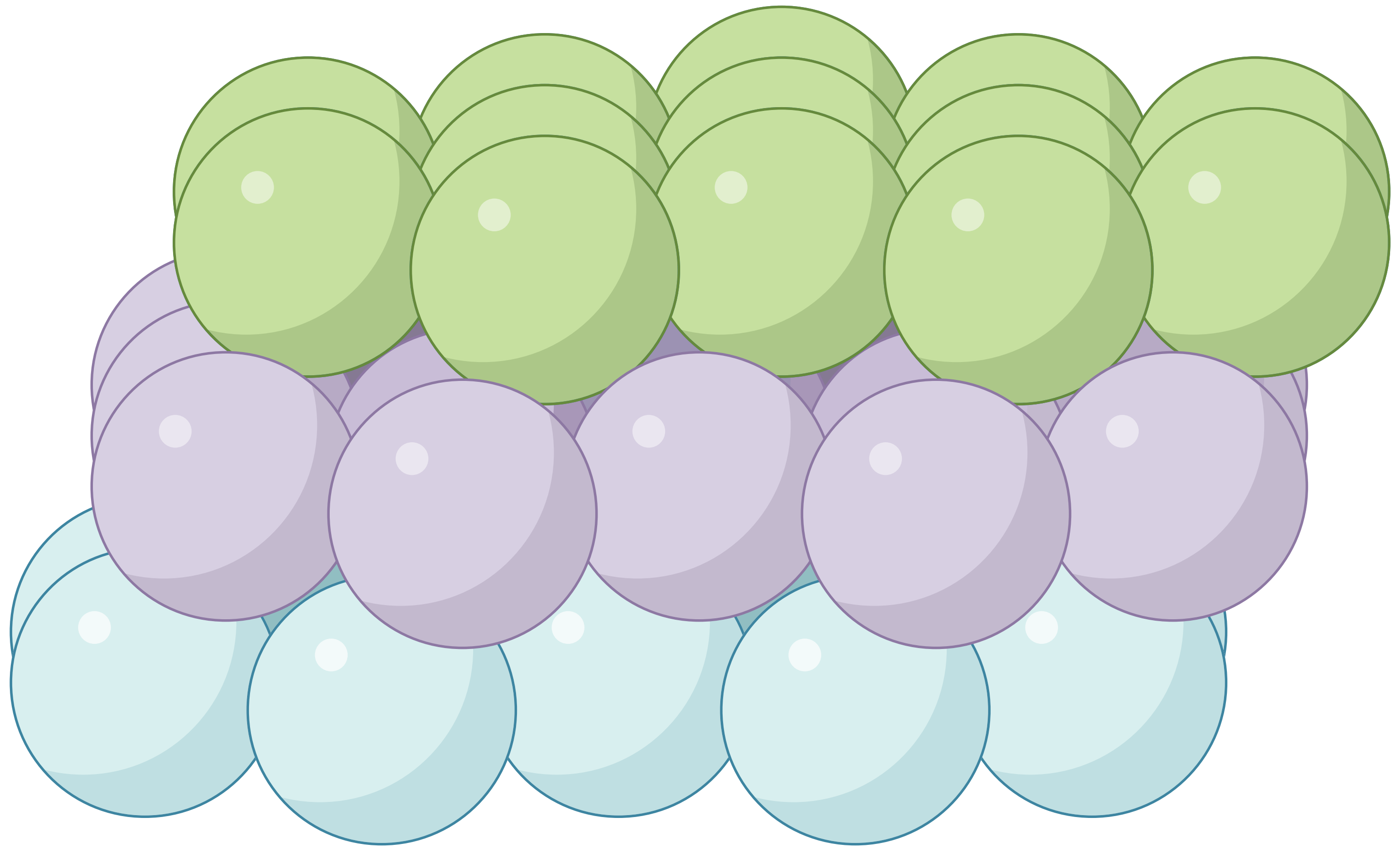
Trivial

$n = 2$



[Thue; 1911]

$n = 3$



[Hales; 1998]

Known for $n = 8$ [Viazovska; 2016] and $n = 24$ [Viazovska et al.; 2016].

Asymptotically the maximal density $\theta(n)$ satisfies:

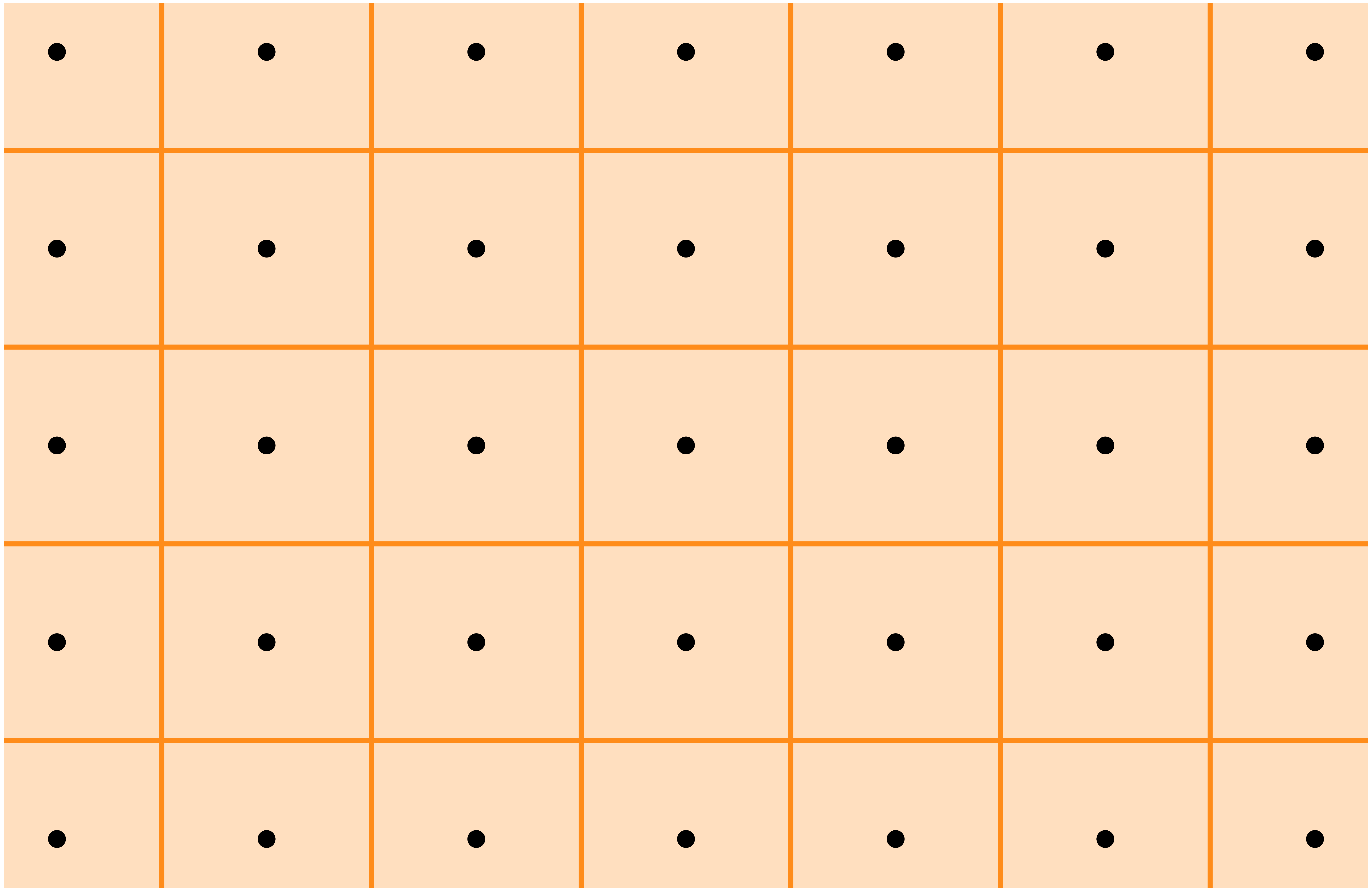
$$c \cdot n^2 \cdot 2^{-n} \leq \theta(n) \leq 2^{-(0.599+o(1))\cdot n}$$

[Klartag; 2025] [Kabatjanskiĭ, Levenšteĭn; 1978]

Question

What is the maximal density of $S \subseteq \mathbb{R}^n$ satisfying $|x-y|_\infty \geq 1$ for distinct $x, y \in S$.

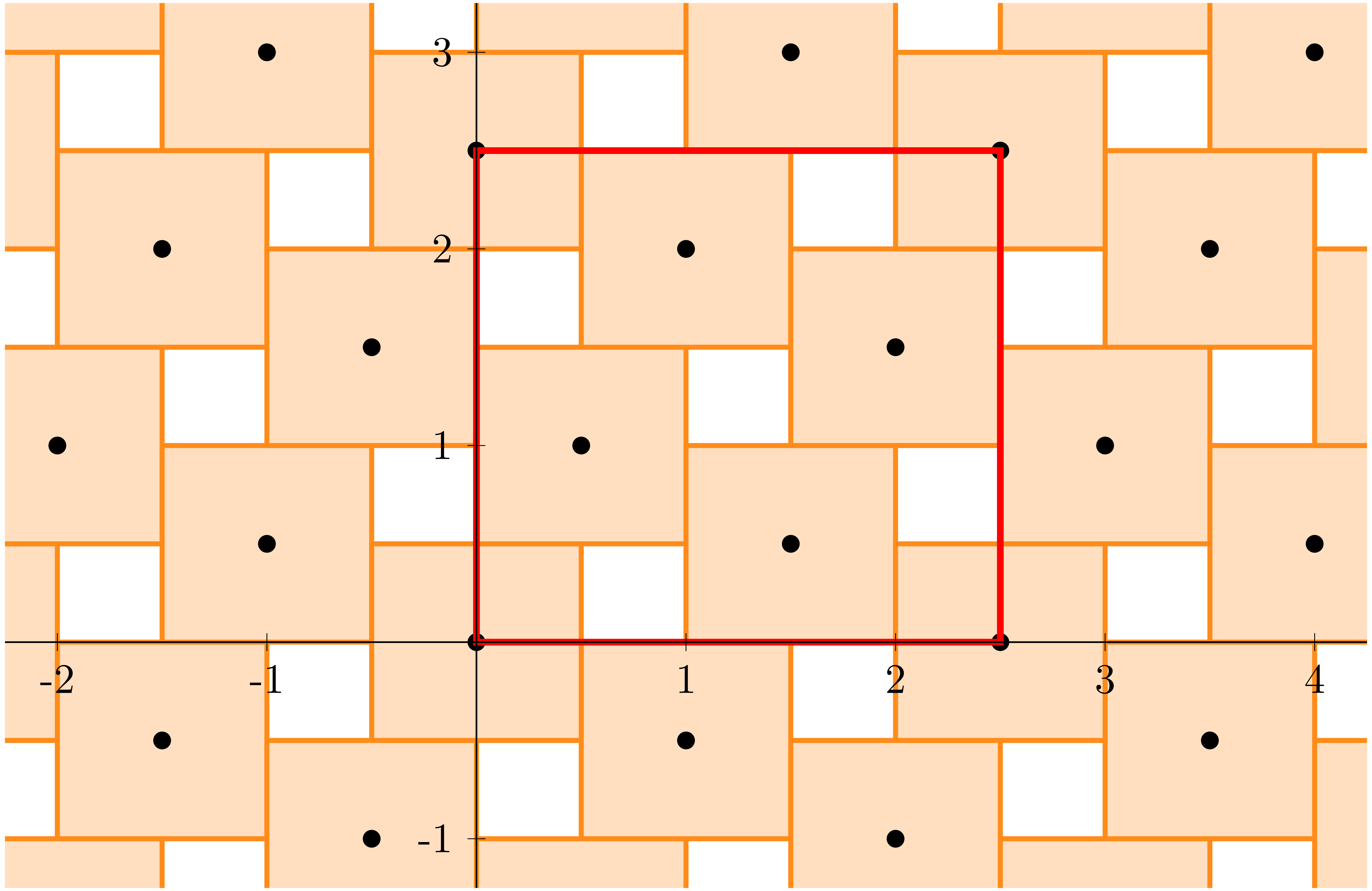
$|x-y|_\infty = \max |x_i - y_i|$



Question

Let $r \geq 1$. What is the maximal size of $S \subseteq \mathbb{R}^n$ satisfying

- $|x - y|_\infty \geq 1$ for distinct $x, y \in S$;
- S is r - periodic in every unit direction.



$r = 5/2$

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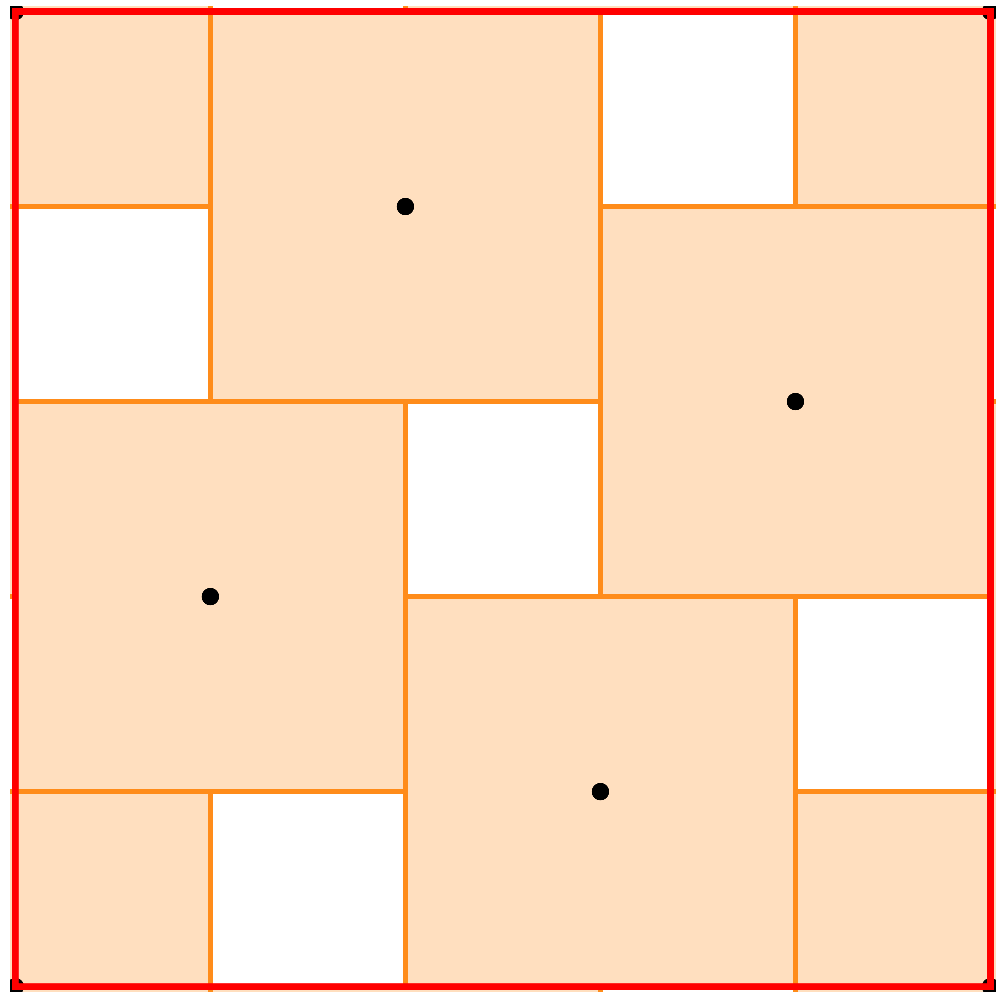
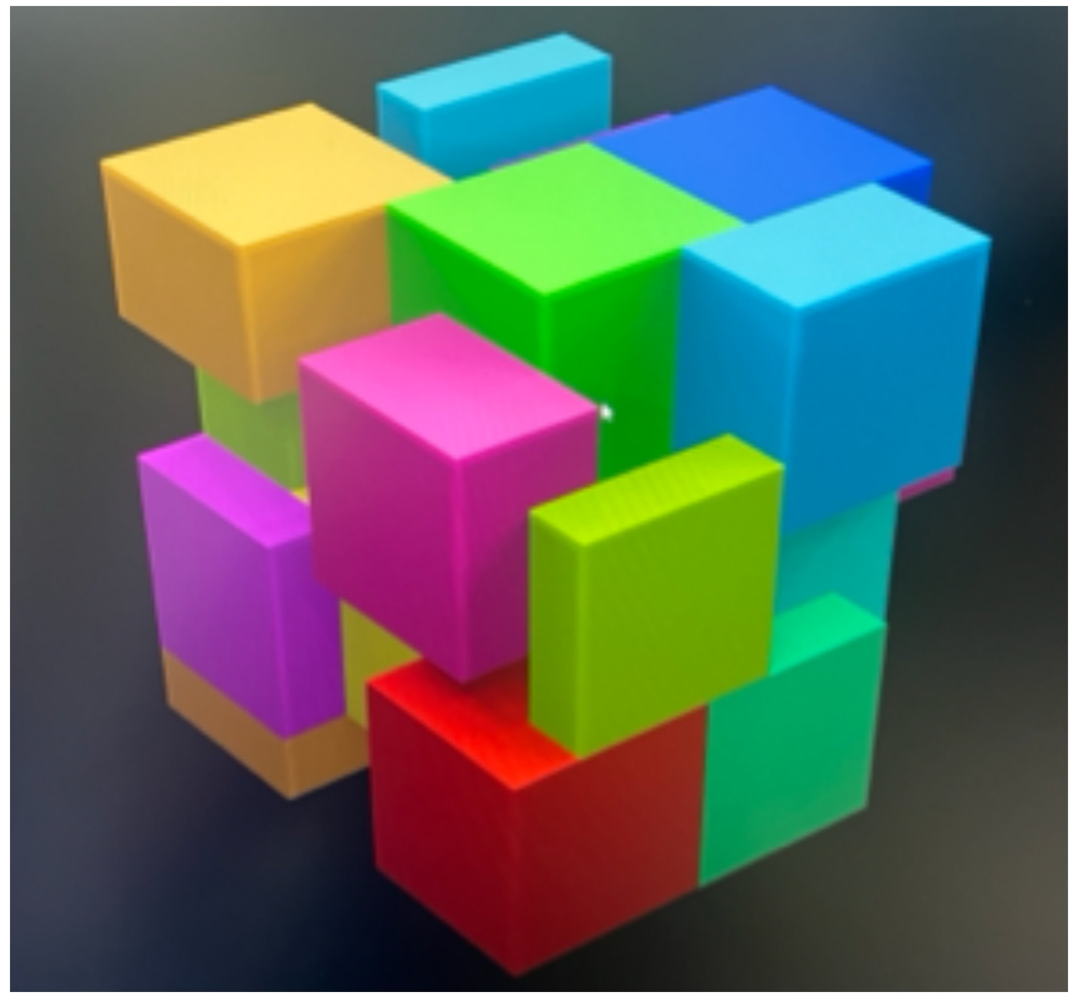


$r = 5/2$



$r = 8/3$

Definition

For $r \in \mathbb{R}_{\geq 1}$ and $n \in \mathbb{N}$ let $\alpha_n(r)$ denote the maximum number of n-dimensional hypercubes of side length one that fit in an n-dimensional torus of side length r.

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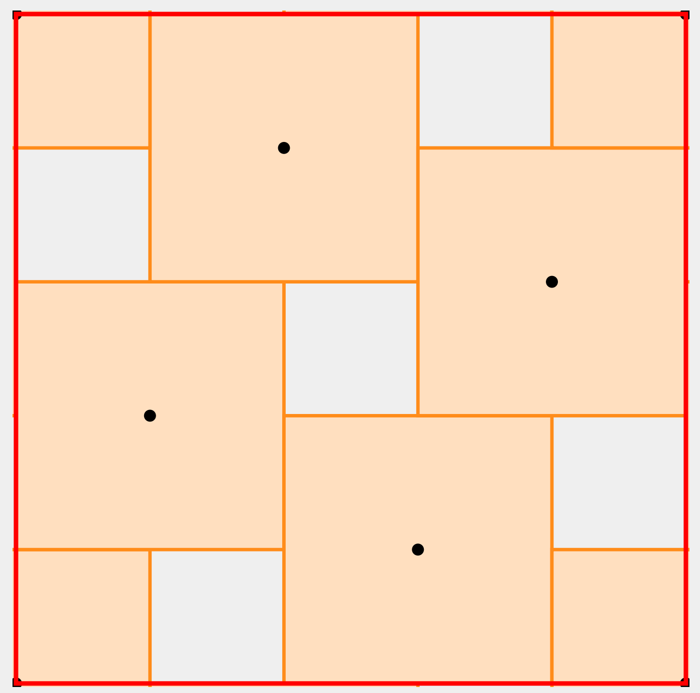
Observations

- The function $r \mapsto \alpha_n(r)$ is nondecreasing.
- For $r \in [1,2)$ we have $\alpha_n(r) = 1$.
- For integer $r \geq 1$ we have $\alpha_n(r) = r^n$.
- The volume of the torus is r^n and thus $\alpha_n(r) \leq r^n$



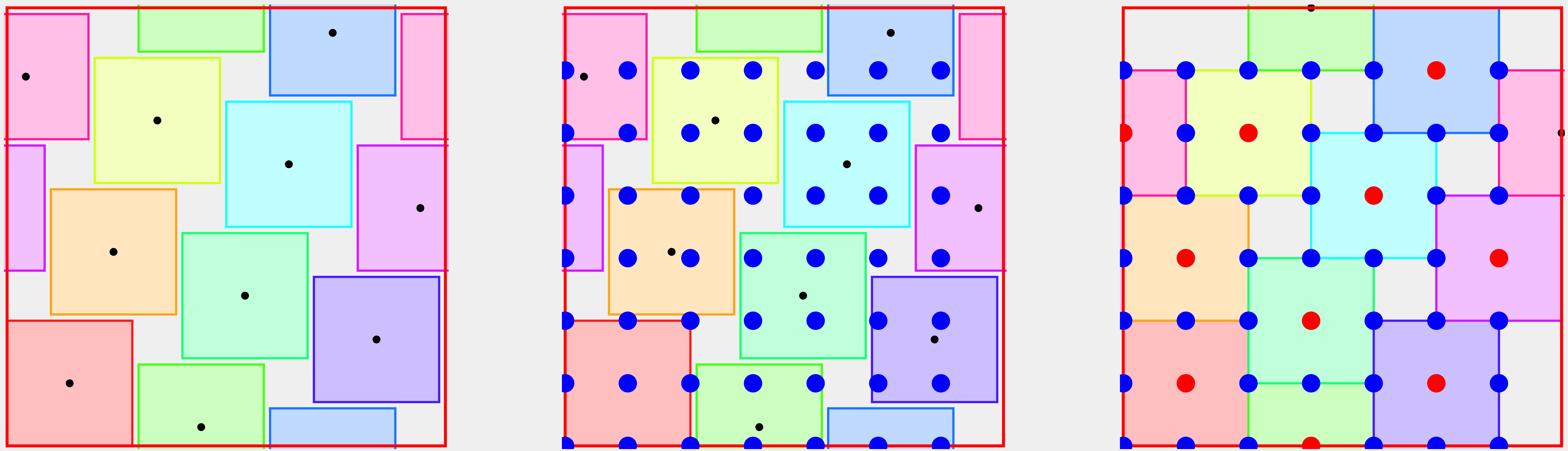
Definition

For $r \in \mathbb{R}_{\geq 1}$ we define:

$$\tilde{\Theta}(r) = \sup_{n \geq 1} \alpha_n(r)^{1/n}$$

Let $r = p/q$ and discretize the n-torus of side length r into a p^n grid.
Any packing can be rounded down to this grid.

Example ($r = 7/2$)

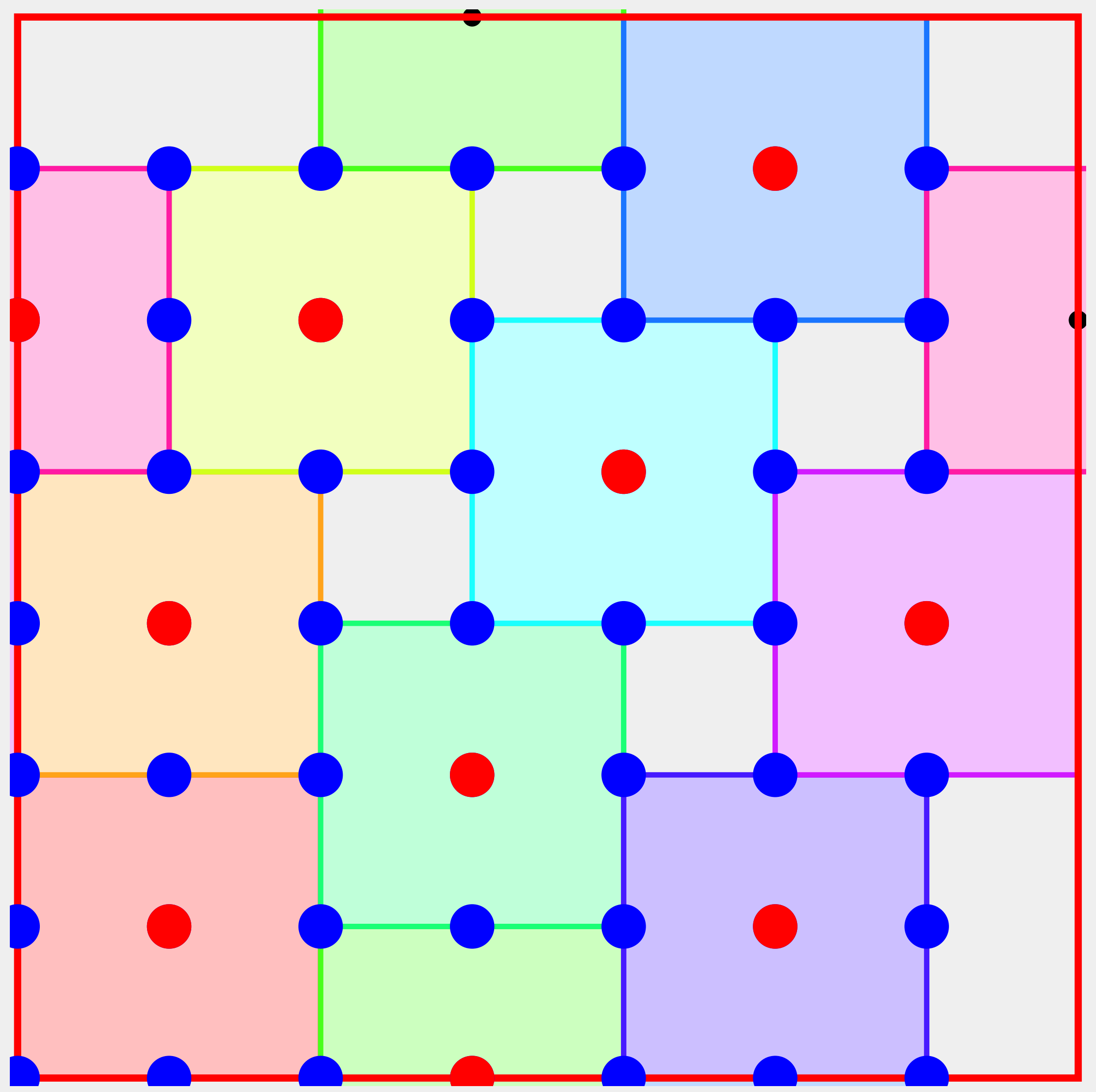


The resulting packing is an independent set in a finite graph.

Definitions

Let $G = (V, E)$ be a graph.

- A set $I \subseteq V$ is called independent if no distinct elements from I are adjacent.
- $\alpha(G)$ is the maximum size of an independent set in G.

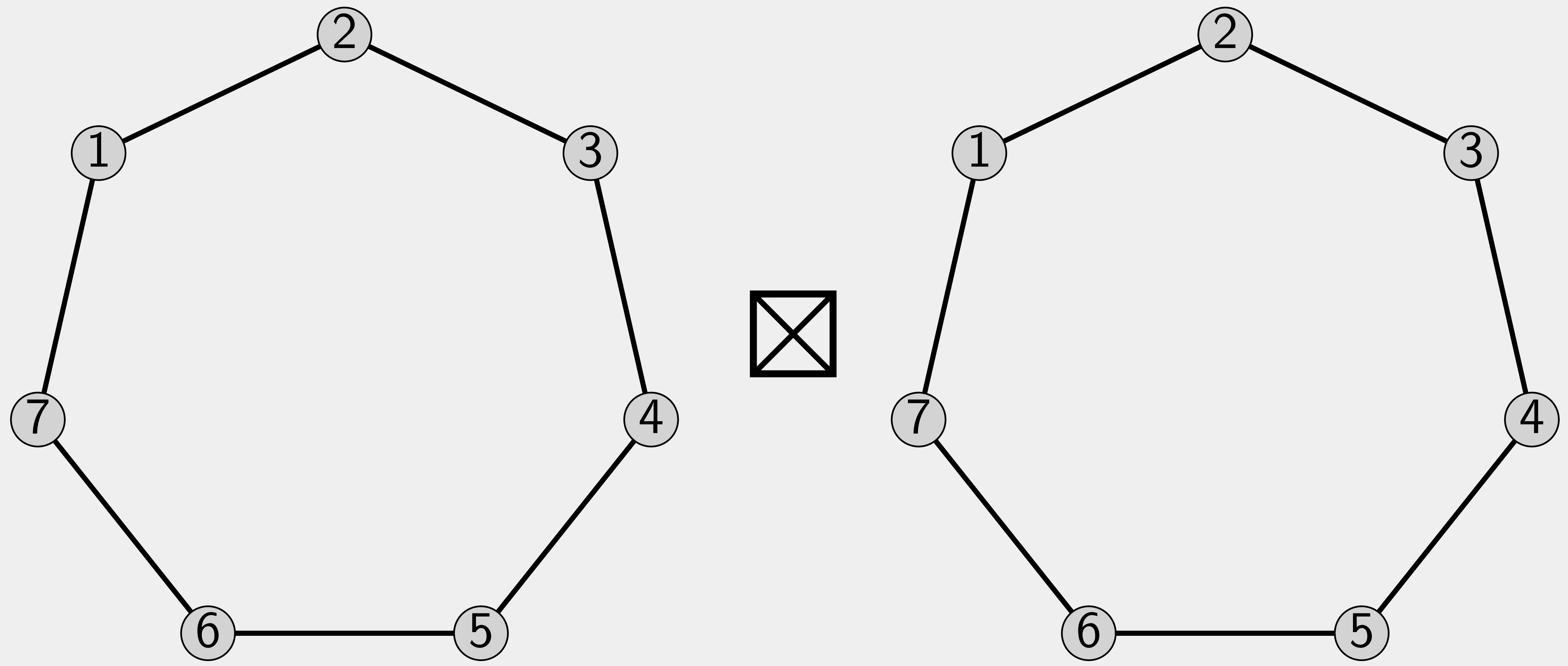


For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the graph $G_1 \boxtimes G_2$ has vertex set $V_1 \boxtimes V_2$ where

$(v_1, v_2) \simeq (u_1, u_2)$ iff $v_1 \simeq u_1$ in G_1 and $v_2 \simeq u_2$ in G_2.

The Shannon Capacity is defined as

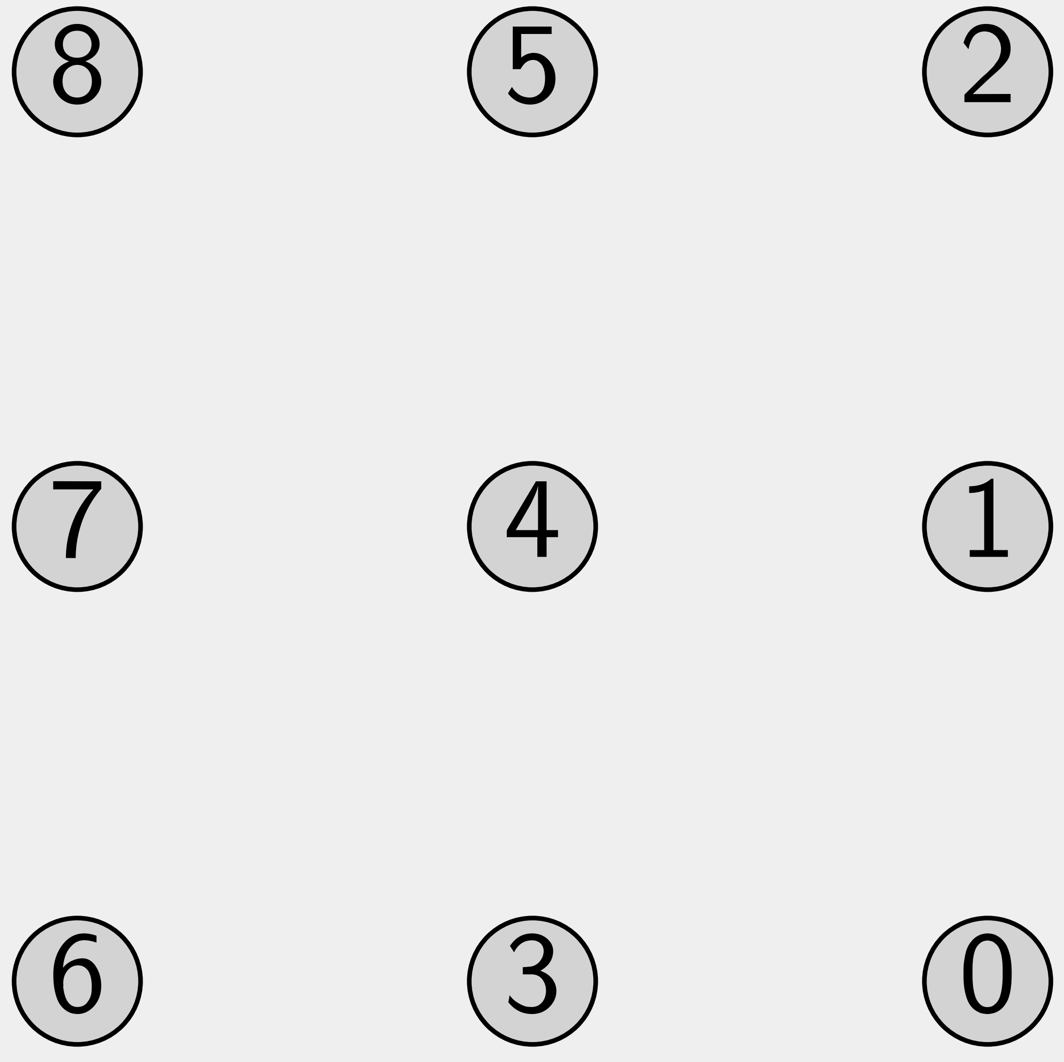
$$\Theta(G) = \sup_{n \geq 1} \alpha(G^{\boxtimes n})^{1/n}$$



We let E_m denote the edgeless graph on m vertices.

So $E_m^{\boxtimes n} = E_{m^n}$ and thus $\alpha(E_m^{\boxtimes n}) = m^n$.

It follows that $\Theta(E_m) = m$.

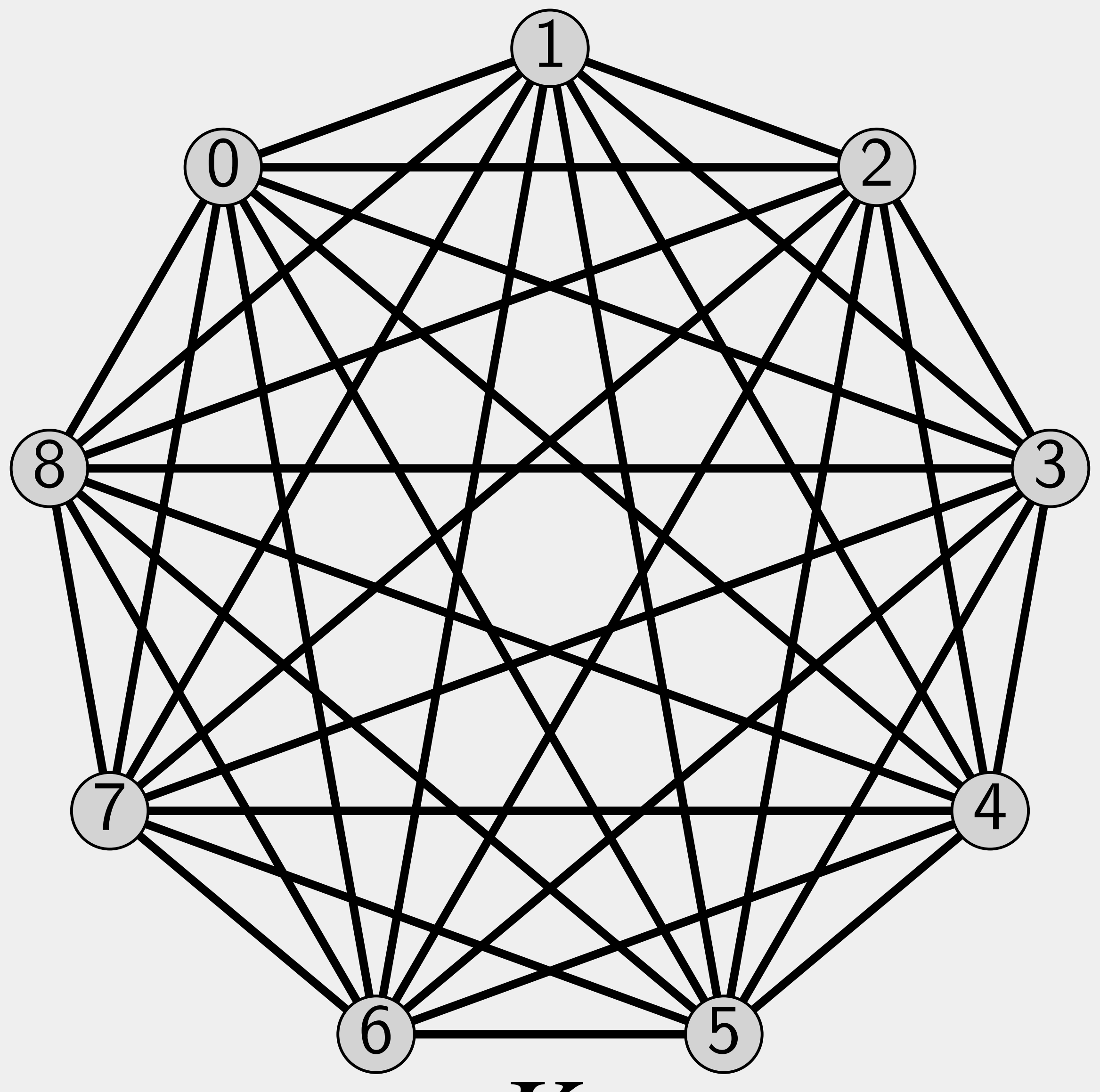


E_9

We let K_m denote the complete graph on m vertices.

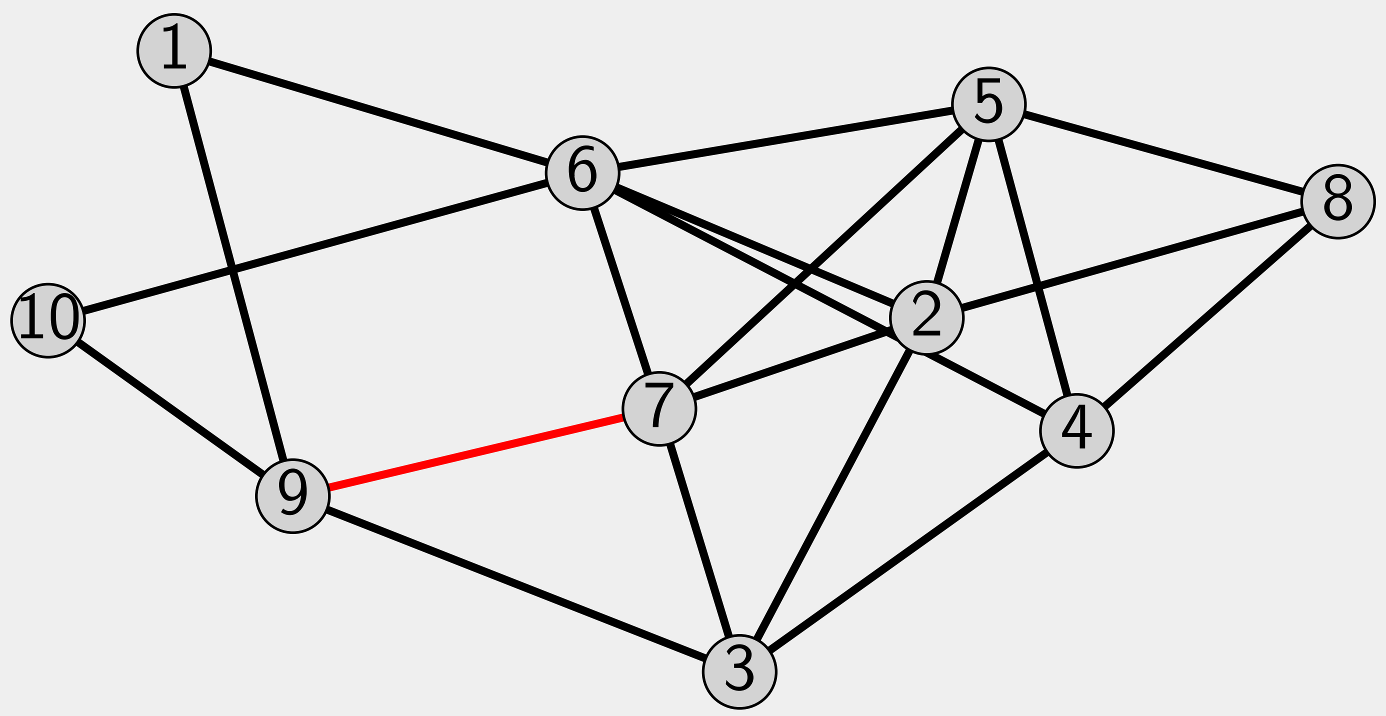
So $K_m^{\boxtimes n} = K_{m^n}$ and thus $\alpha(K_m^{\boxtimes n}) = 1$.

It follows that $\Theta(K_n) = 1$.

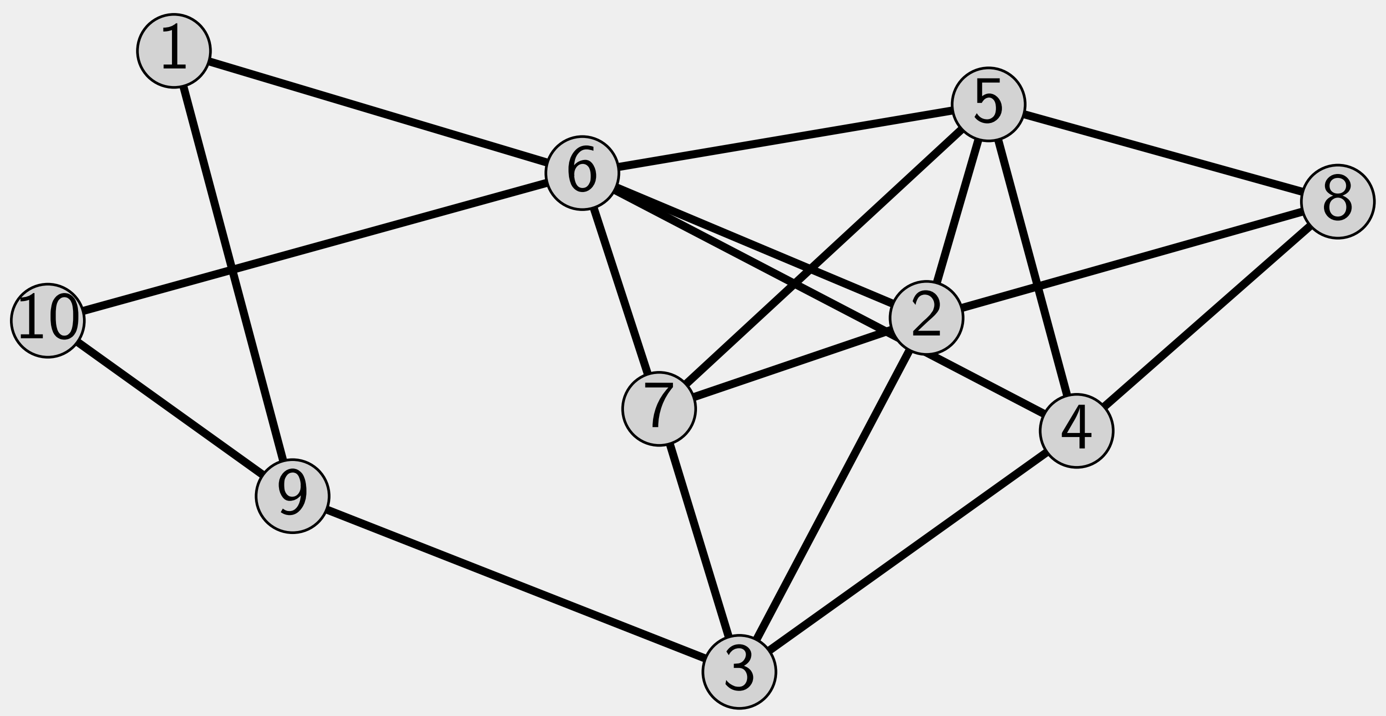


K_9

If H is obtained by removing an edge from G, then $\Theta(G) \leq \Theta(H)$:

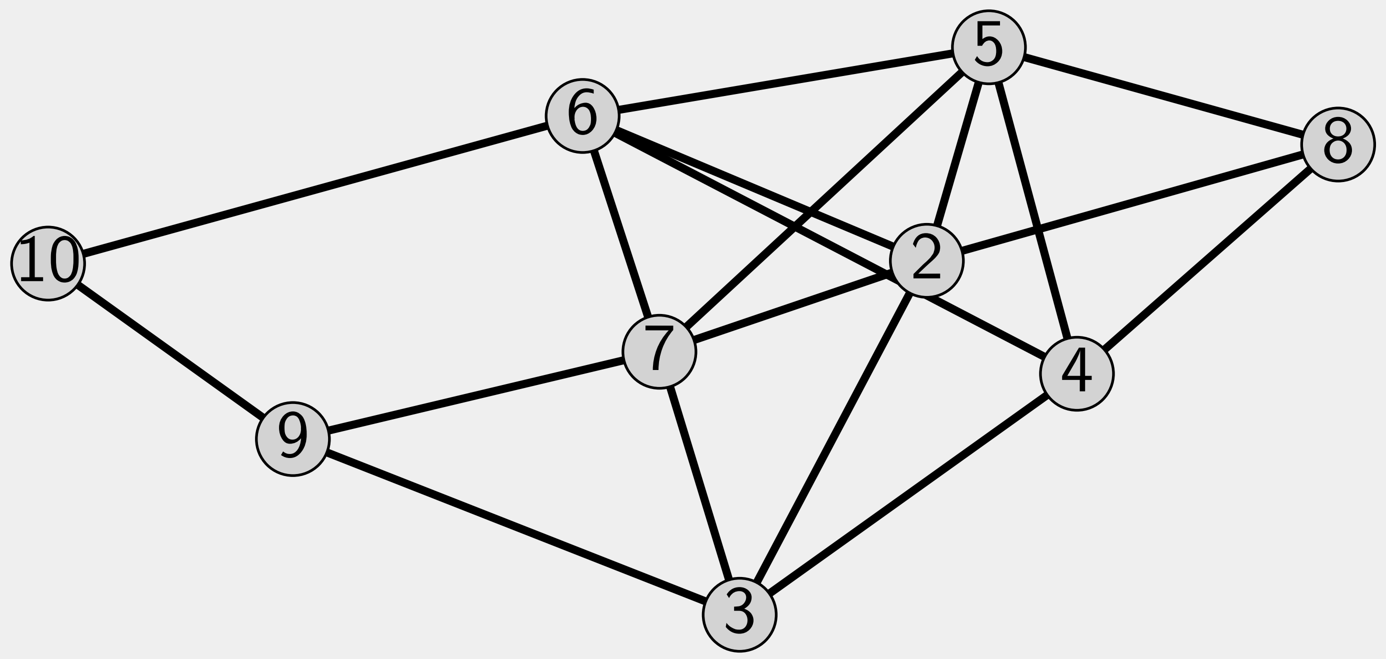


G

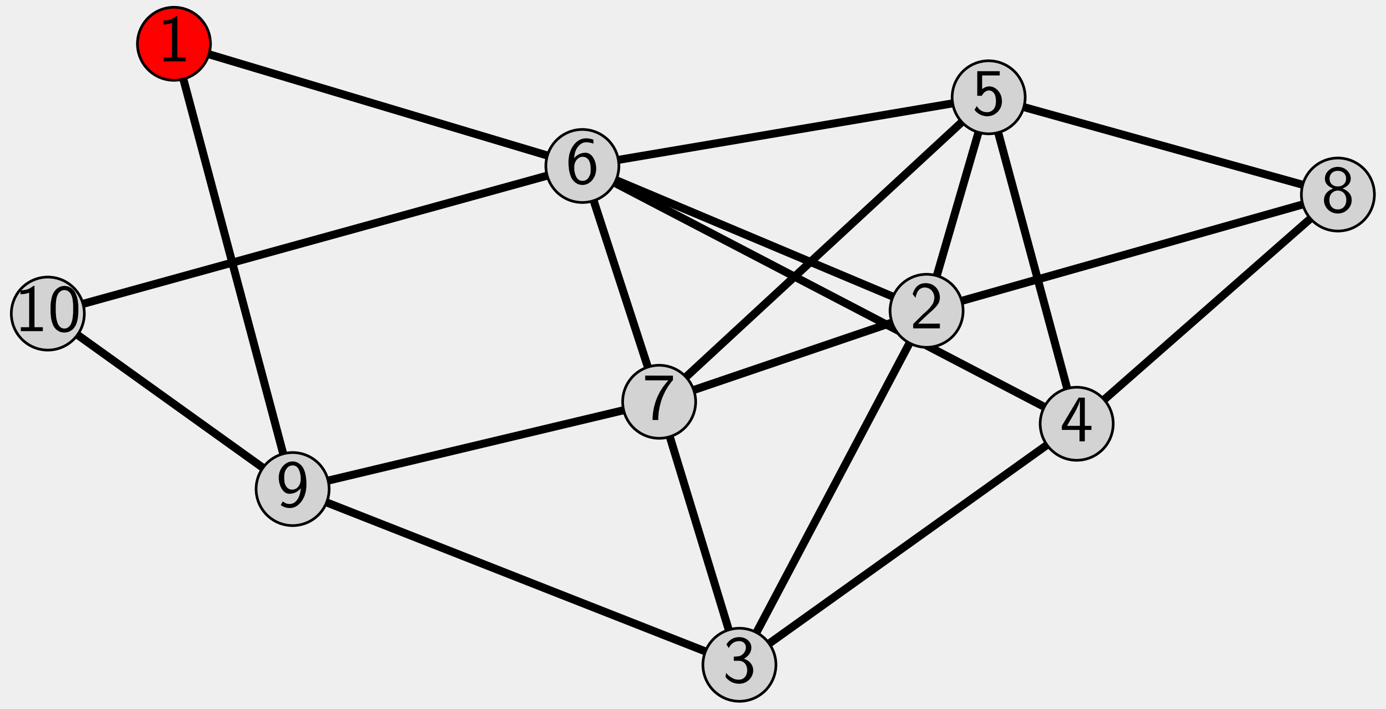


H

If G is obtained by removing a vertex from H, then $\Theta(G) \leq \Theta(H)$:



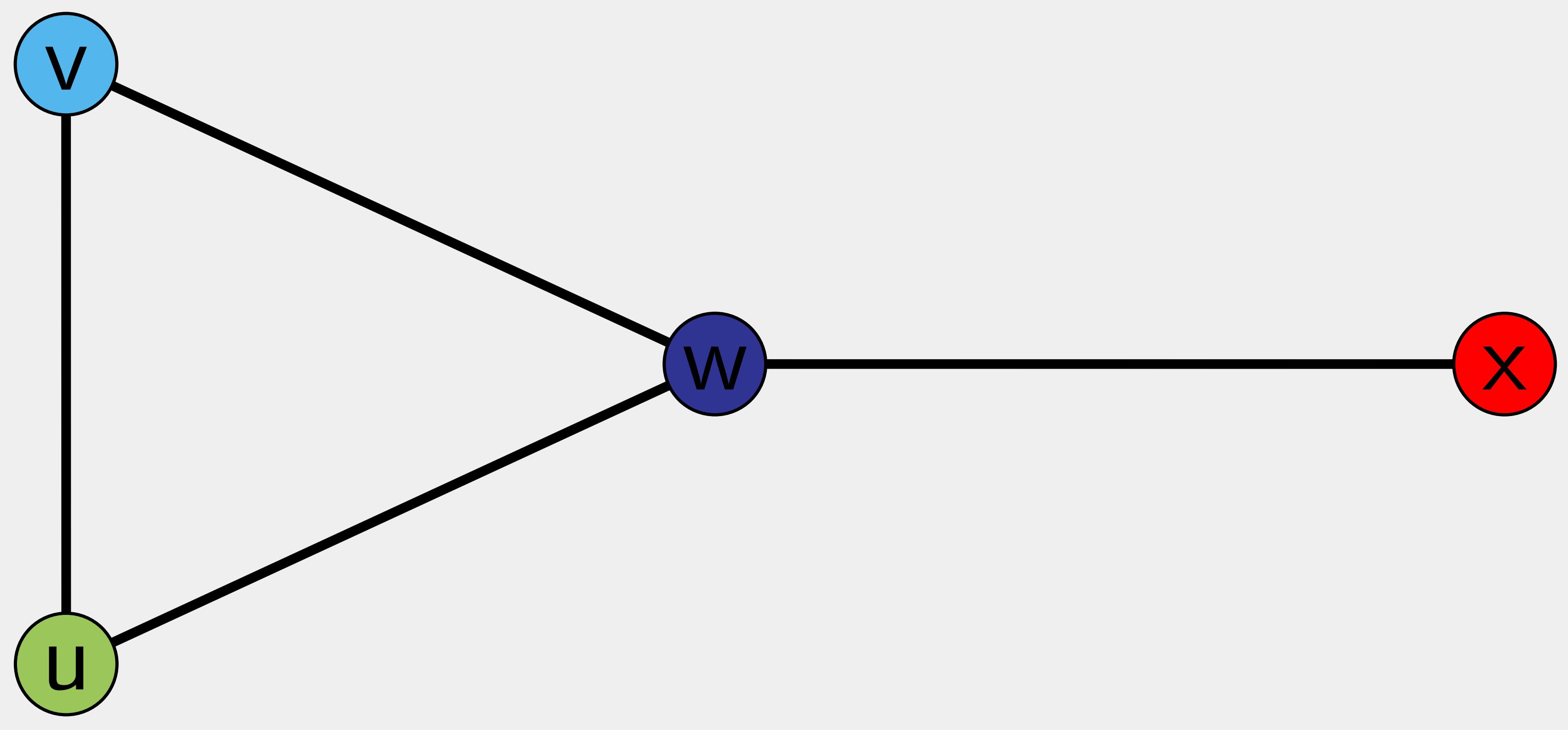
G



H

In both cases write $G \leq H$.

Suppose there is a pair of vertices u, v such that

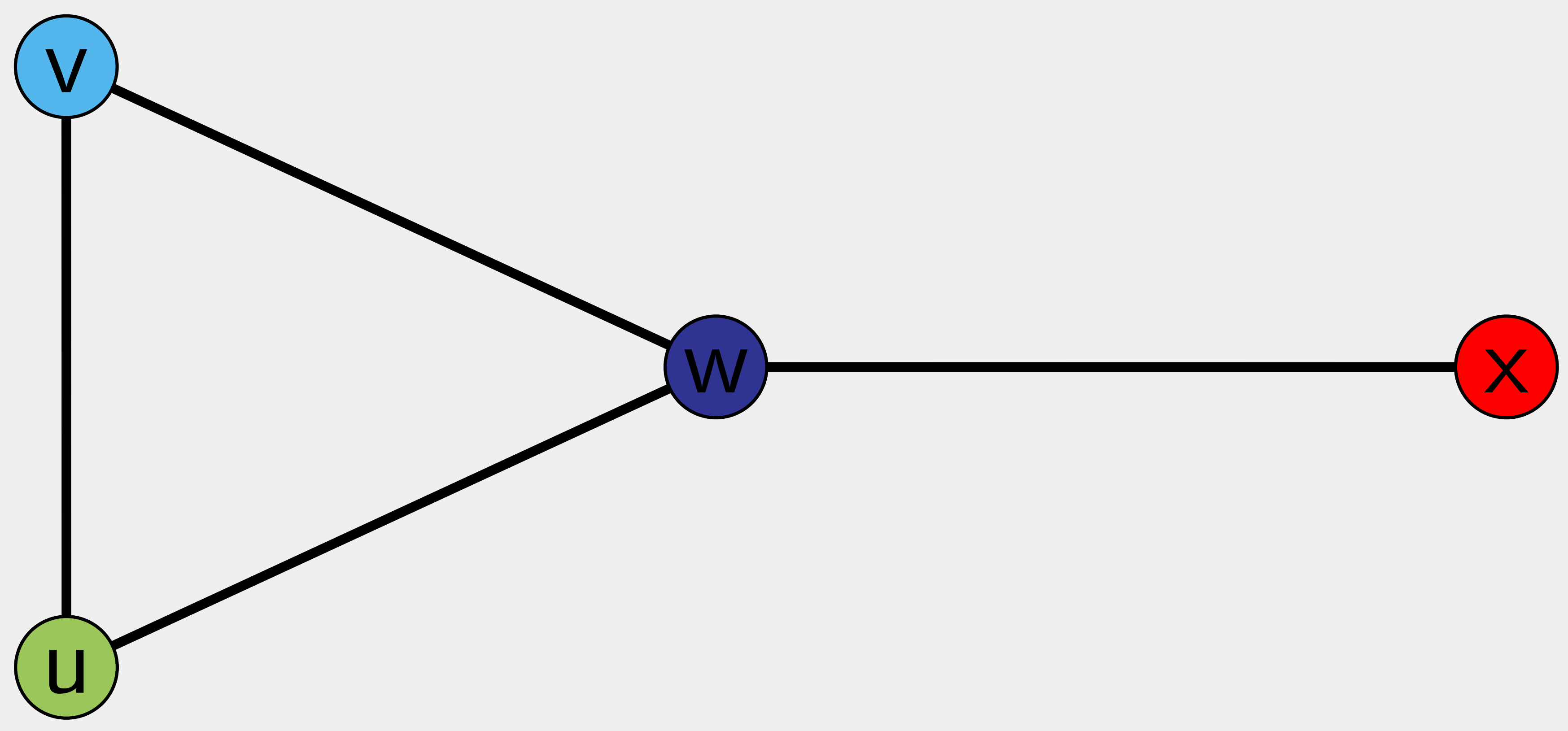


- $u \sim v$
- $u \sim a$ iff $v \sim a$ for all a

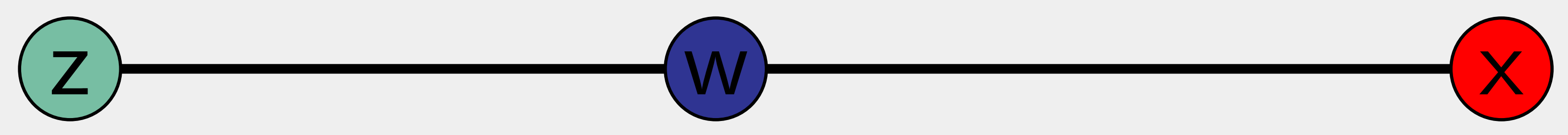
In any independent set of $G^{\boxtimes n}$ we may collapse any occurrence of v and u.

Example:

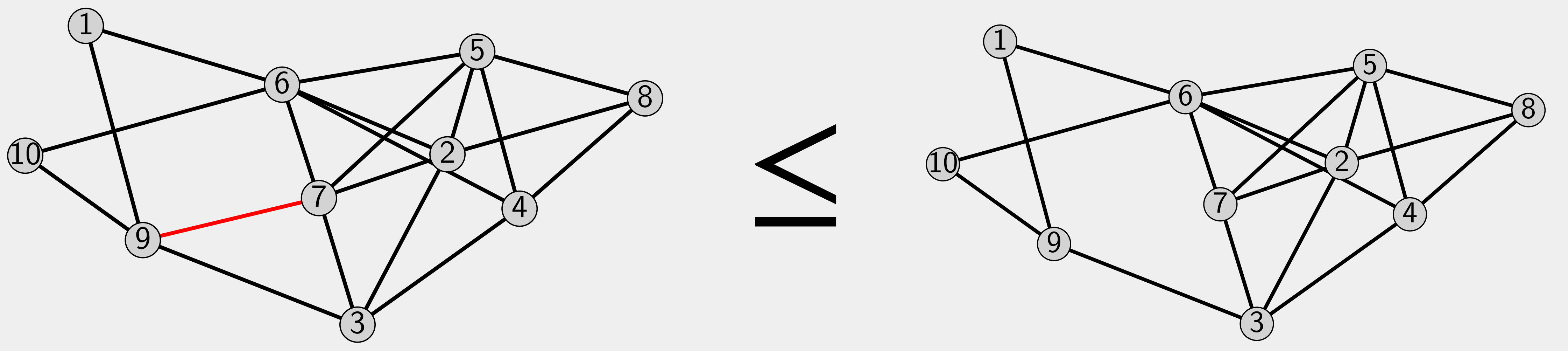
$$\{vx, xu, vu, xx\}$$
$$\{zx, xz, zz, xx\}$$



$\leq$

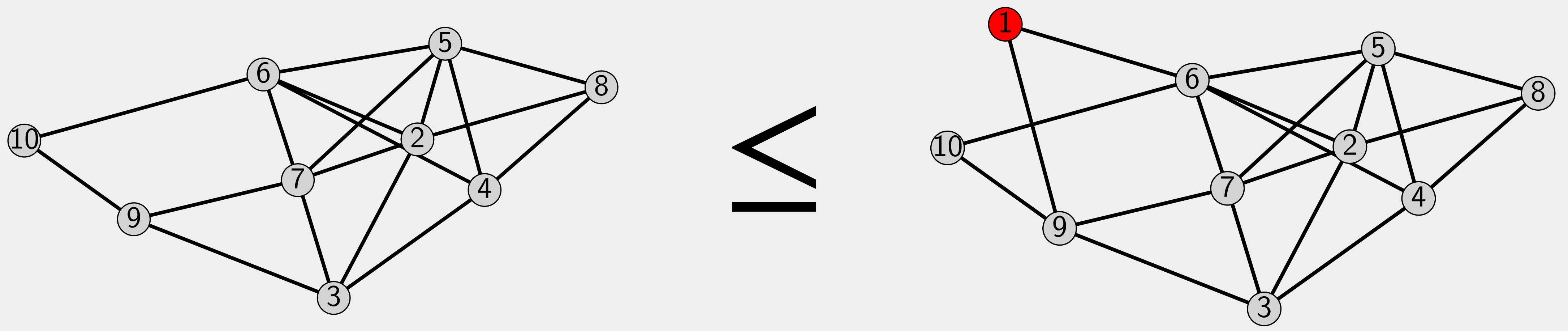


We define the preorder $\leq$ on the set of graphs $\mathscr{G}$ as the transitive closure.



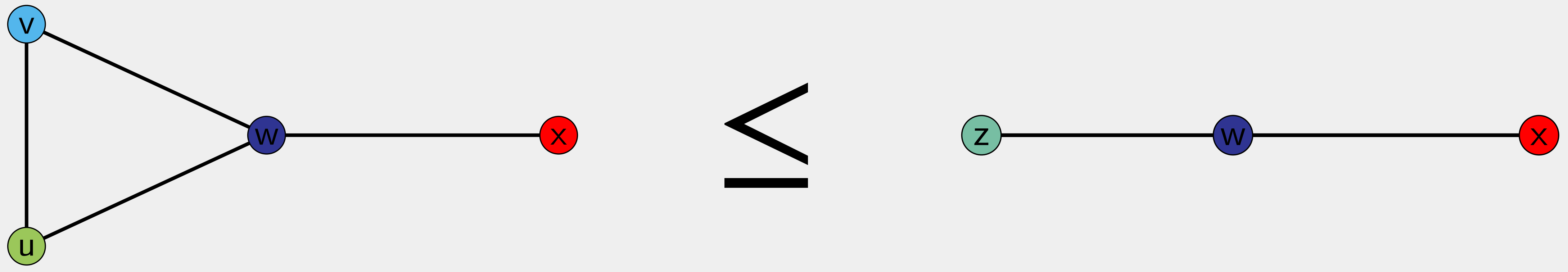
Terminology:

$G \leq H$ iff there is a cohomomorphism from G to H



Observations:

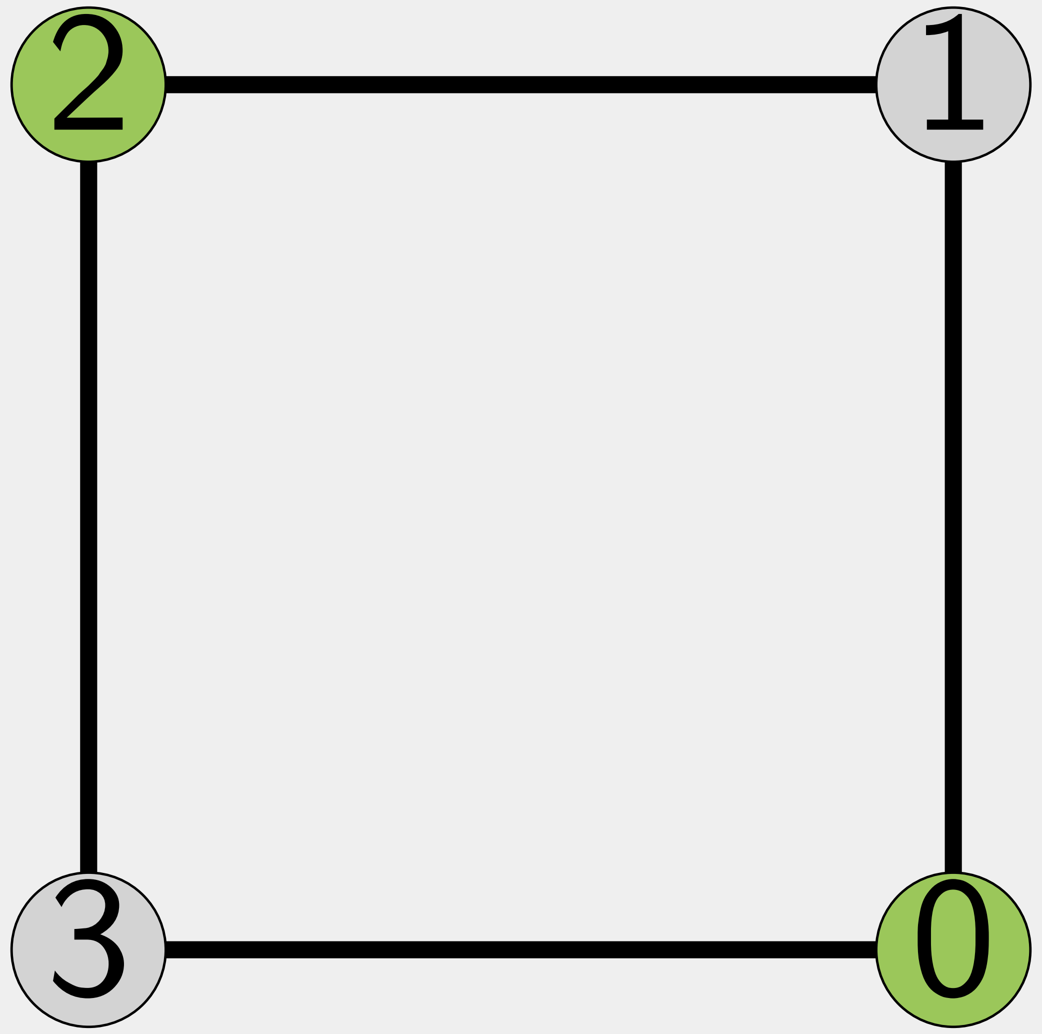
If $G \leq H$ then $\alpha(G) \leq \alpha(H)$ and $\Theta(G) \leq \Theta(H)$.



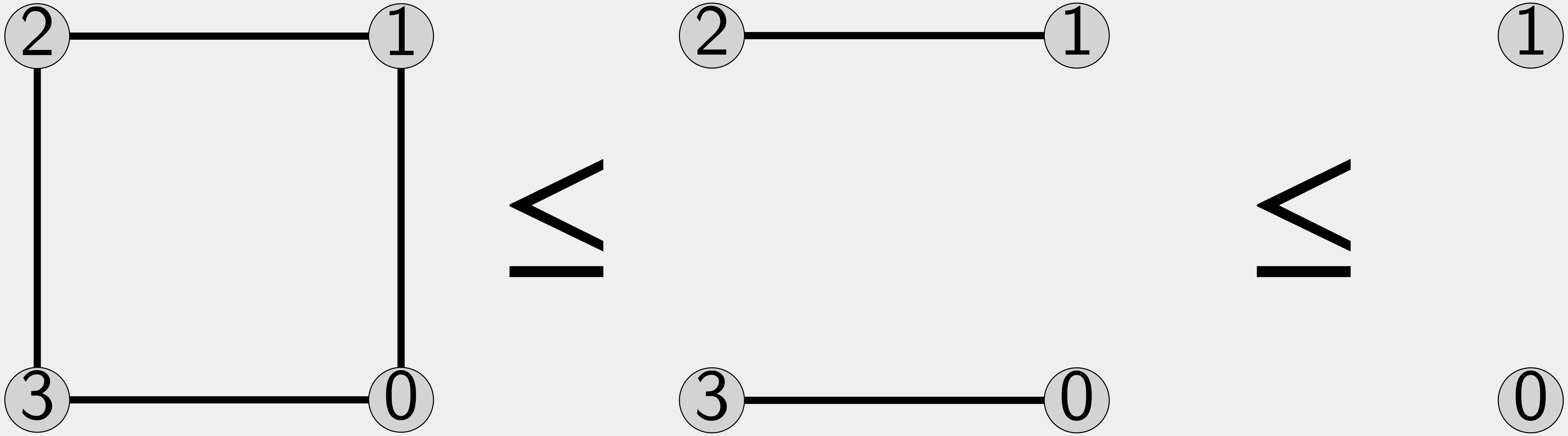
We have:

$$\alpha(G) = \max\{n \in \mathbb{N} \,|\, E_n \leq G\}$$

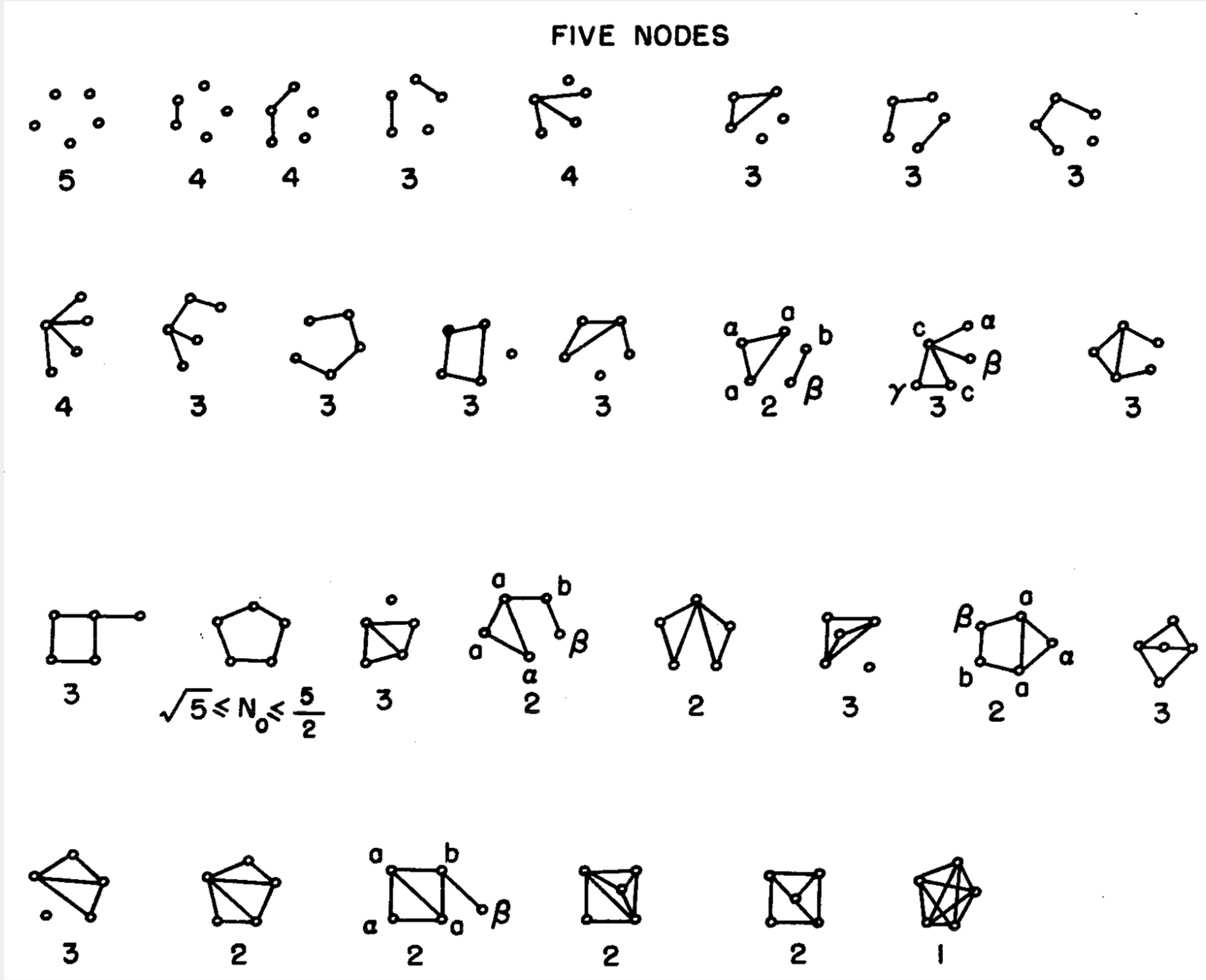
Question. What is $\Theta(C_4)$?



We have $\alpha(C_4) = 2$ and thus $\Theta(C_4) \geq 2$. Also:



And thus $C_4 \leq E_2$. It follows that $\Theta(C_4) = 2$.



Shannon 1956

Definition

The Shannon Capacity is defined as

$$\Theta(G) = \sup_{n \geq 1} \alpha(G^{\boxtimes n})^{1/n}$$

Suppose $f : \mathscr{G} \to \mathbb{R}_{\geq 0}$ is a graph parameter that satisfies:

- $\alpha(G) \leq f(G)$ for all $G \in \mathscr{G}$
- $f(G \boxtimes H) = f(G) \boxtimes f(H)$ for all $G, H \in \mathscr{G}$

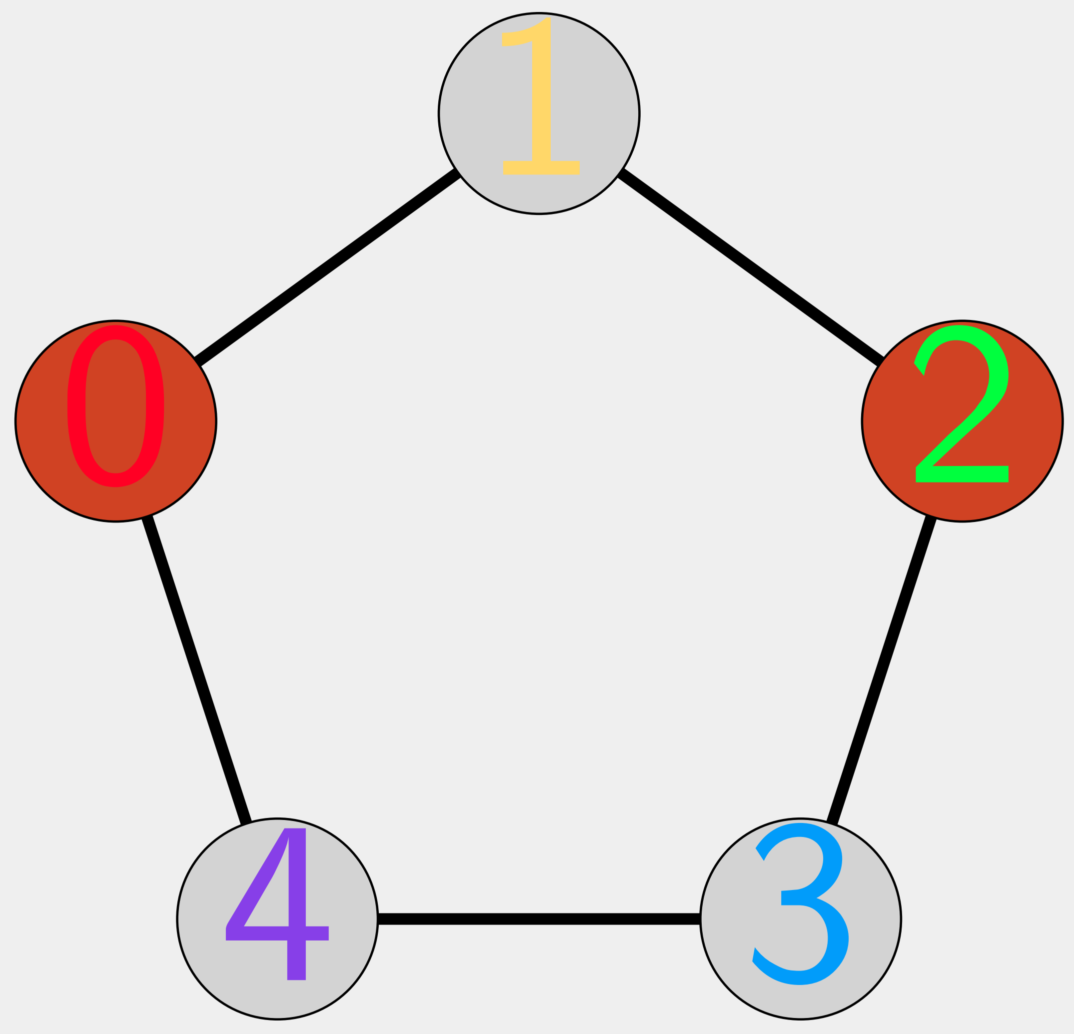
Then:

$$\Theta(G) = \sup_{k} \alpha(G^{\boxtimes k})^{1/k} \leq \sup_{k} f(G^{\boxtimes k})^{1/k} = f(G).$$

Consider a graph G with vertex set V and independent set $I \subseteq V$.

Let $\vec{v} \in \mathbb{R}^V$ be the characteristic vector of I and consider the matrix $M = \frac{1}{|I|}\vec{v}\vec{v}^T$

Example: $\vec{v} = (1{,}0{,}1{,}0{,}0)$ $M = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$



1. The matrix M is symmetric and positive semi-definite.
2. If u and v are adjacent in G then $M_{uv} = 0$.
3. $\mathrm{tr}(M) = 1$
4. $\sum_{v,u \in V} M_{u,v} = |I|$.

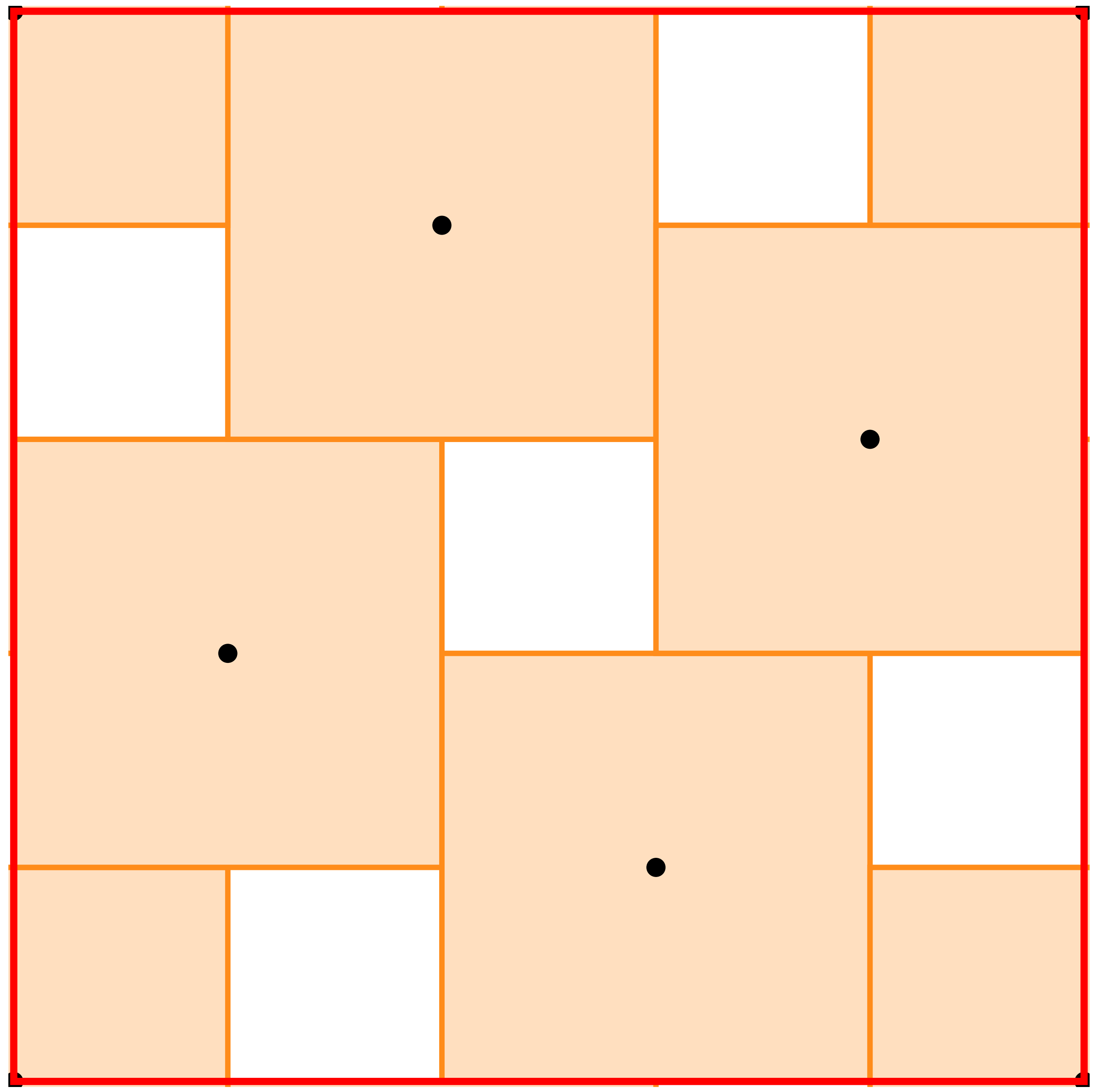
The Lovász theta function $\vartheta(G)$ is defined as $\max_M \sum_{v,u \in V} M_{u,v}$ taken over M satisfying 1-3.

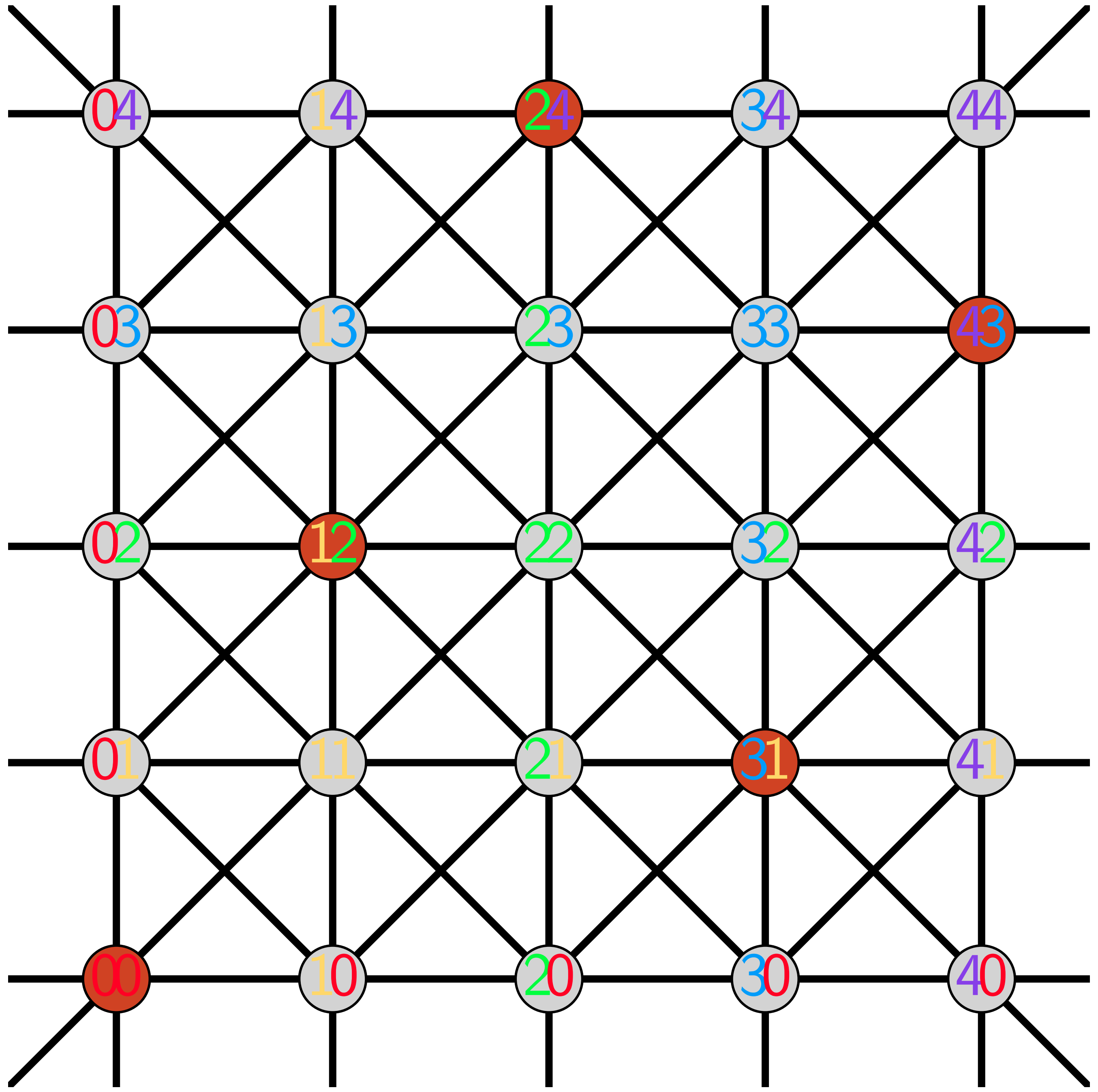
Theorem (Lovász 1979)

We have $\alpha(G) \leq \vartheta(G)$ and $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$ and thus $\Theta(G) \leq \vartheta(G)$.

Corollary

We have $\Theta(C_5) \leq \vartheta(C_5) = \sqrt{5}$ and thus $\Theta(C_5) = \sqrt{5}$.





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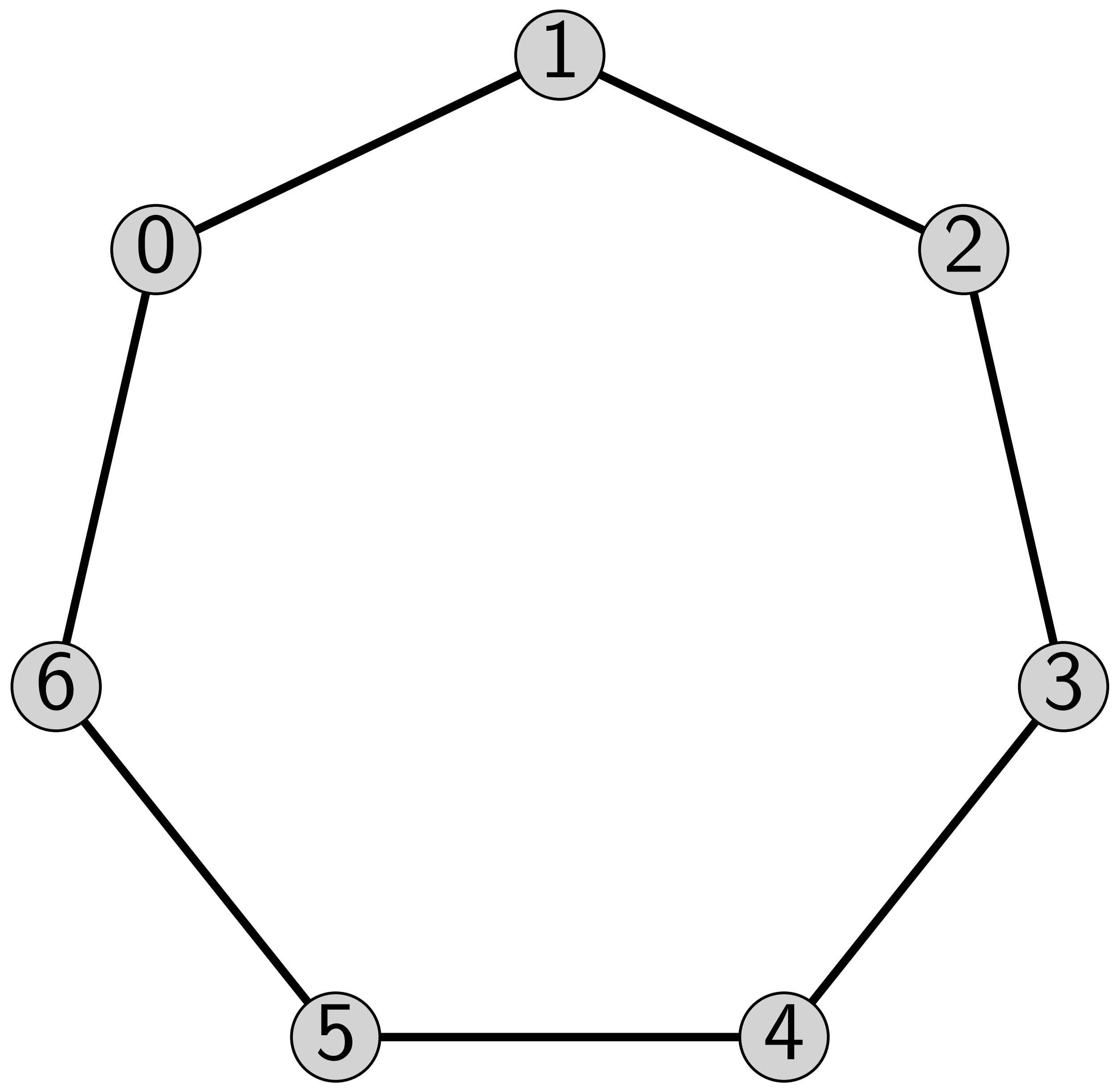
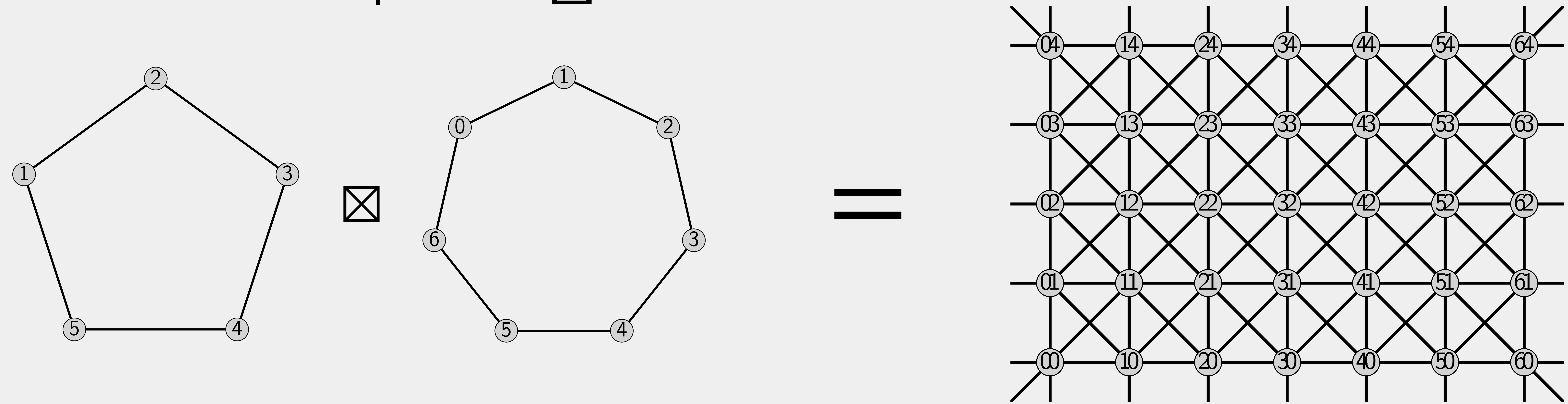
Open Question:

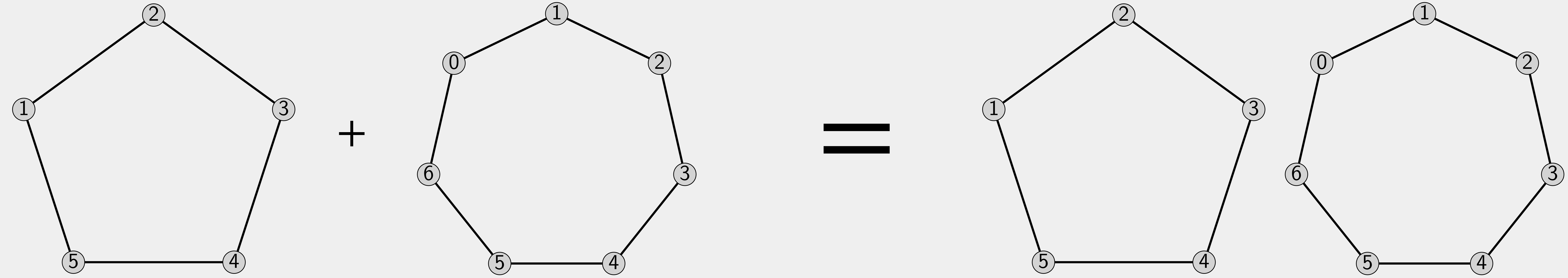
What is $\Theta(C_7)$?



Recall we defined multiplication ⊠:



We also define addition +:



The space of graphs $\mathscr{G}$ is a Strassen preordered semiring with

- Multiplication $\boxtimes$
- Addition $+$
- Preorder $\leq$
- A subsemiring isomorphic to $\mathbb{N}$ given by $\{E_0, E_1, E_2, \dots\}$

Strassen semiring (original) example (1986–1991)

The space of (three-fold) tensors

- Multiplication: Tensor product $\otimes$
- Addition: Direct sum $\oplus$
- Preorder: $T \leq T'$ iff there are linear maps A_1, A_2, A_3 such that $T = (A_1, A_2, A_3)T'$
- A subsemiring isomorphic to $\mathbb{N}$ given by $\{I_0, I_1, I_2, \dots\}$

The space of graphs $\mathscr{G}$ is a Strassen preordered semiring with

- Multiplication $\boxtimes$
- Addition $+$
- Preorder $\leq$
- A subsemiring isomorphic to $\mathbb{N}$ given by $\{E_0, E_1, E_2, \ldots\}$

The subrank of $G \in \mathscr{G}$ is defined as $Q(G) := \max\{n \in \mathbb{N} \mid E_n \leq G\}$.

The asymptotic subrank is defined as $Q_{\sim}(G) := \lim_{k \to \infty} Q(G^k)^{1/k}$.

For graphs $Q(G) = \alpha(G)$ and thus $Q_{\sim}(G) = \Theta(G)$.

The space of graphs $\mathcal{G}$ is a Strassen semiring with

- Multiplication $\boxtimes$
- Addition $+$
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- A subsemiring isomorphic to $\mathbb{N}$ given by $\{E_0, E_1, E_2, \ldots\}$

The asymptotic spectrum $\mathbf{X}(\mathcal{G})$ is the set of functions $f : \mathcal{G} \to \mathbb{R}_{\geq 0}$ that are

- Multiplicative: $f(G \boxtimes H) = f(G)f(H)$;
- Additive: $f(G + H) = f(G) + f(H)$;
- Monotone: $G \leq H$ implies $f(G) \leq f(H)$;
- Normalized: $f(E_n) = n$.

Example: The Lovász theta function ϑ is a member of $\mathbf{X}(\mathcal{G})$.

The asymptotic spectrum $\mathbf{X}(\mathscr{G})$ is the set of functions $f : \mathscr{G} \to \mathbb{R}_{\geq 0}$ that are

- Multiplicative: $f(G \boxtimes H) = f(G)f(H)$;
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Subrank $Q(G) := \max\{n \in \mathbb{N} \mid E_n \leq G\}$ — Asymptotic Subrank $\underset{\sim}{Q}(G) := \lim_{k\to\infty} Q(G^k)^{1/k}$

Theorem (Strassen 1988)

If a is an element of a Strassen semiring R then $\underset{\sim}{Q}(a) = \min_{f \in \mathbf{X}(R)} f(a)$.

Corollary (Zuiddam 2018)

For graphs $G \in \mathscr{G}$ we have $\Theta(G) = \min_{f \in \mathbf{X}(\mathscr{G})} f(G)$.

We define a metric on $\mathscr{G}$:

$$d(G,H) = \max_{f \in \mathbf{X}(\mathscr{G})} |f(G) - f(H)|.$$

If G_n converges to G then $\Theta(G_n)$ converges to $\Theta(G)$.

Theorem (Polak 2019)

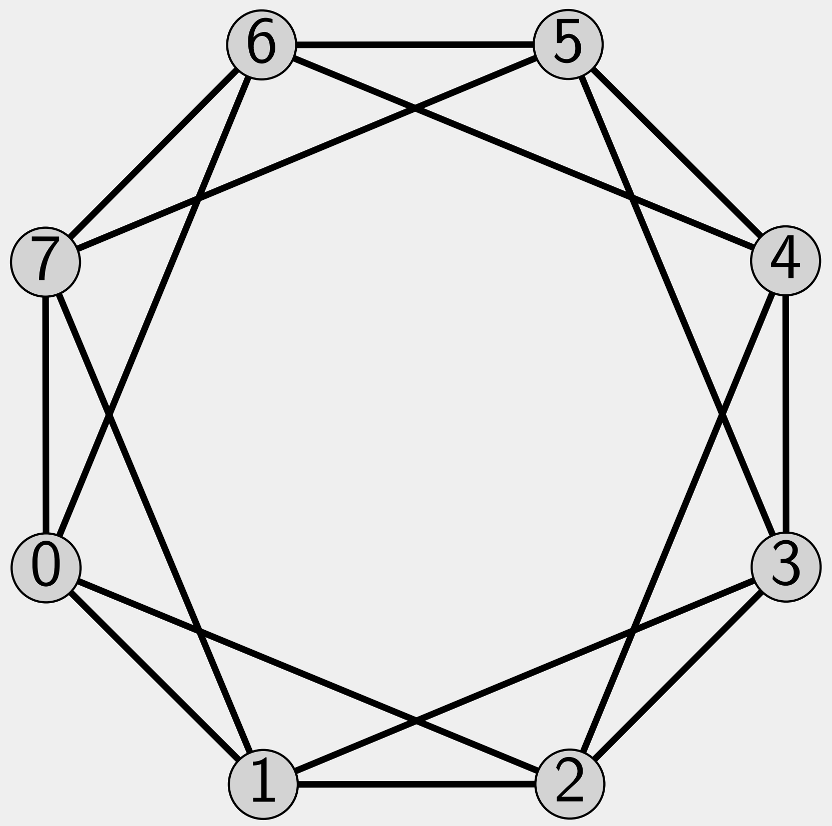
For $n \geq 3$ there are nontrivial sequences approaching E_n from above and from below.

Theorem (De Boer, B., Zuiddam 2019)

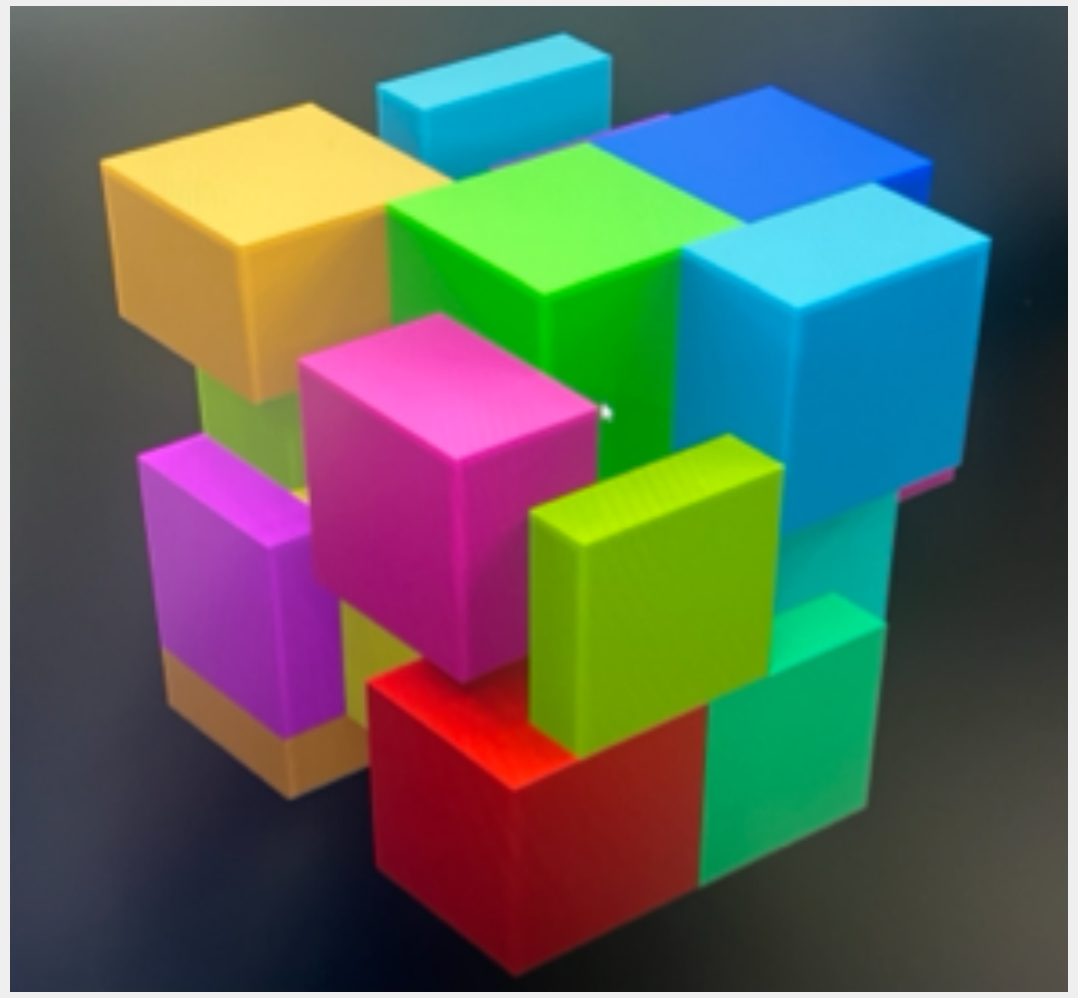
- For $n \geq 3$ odd there are nontrivial sequences approaching C_n and $\overline{C_n}$ from above.
- There are Cauchy sequence G_n with no limit in $\mathscr{G}$, i.e. $\mathscr{G}$ is not complete.

For $2 \le p/q$ define the graph $E_{p/q}$ with

- Vertex set: $\mathbb{Z}_p$
- Edge set: $x \sim y$ iff $-q < (x - y) \mod p < q$.



$E_{8/3}$



$r = 8/3$

Recall: $\tilde{\Theta}(p/q) = \Theta(E_{p/q})$

For $2 \leq p/q$ define the graph $E_{p/q}$ with

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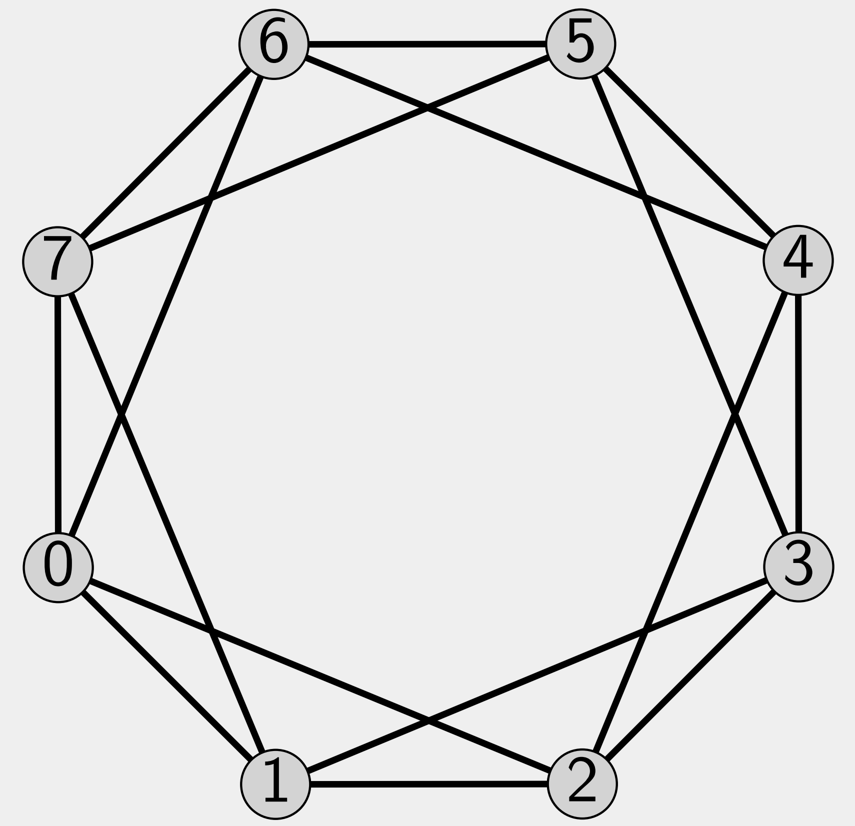
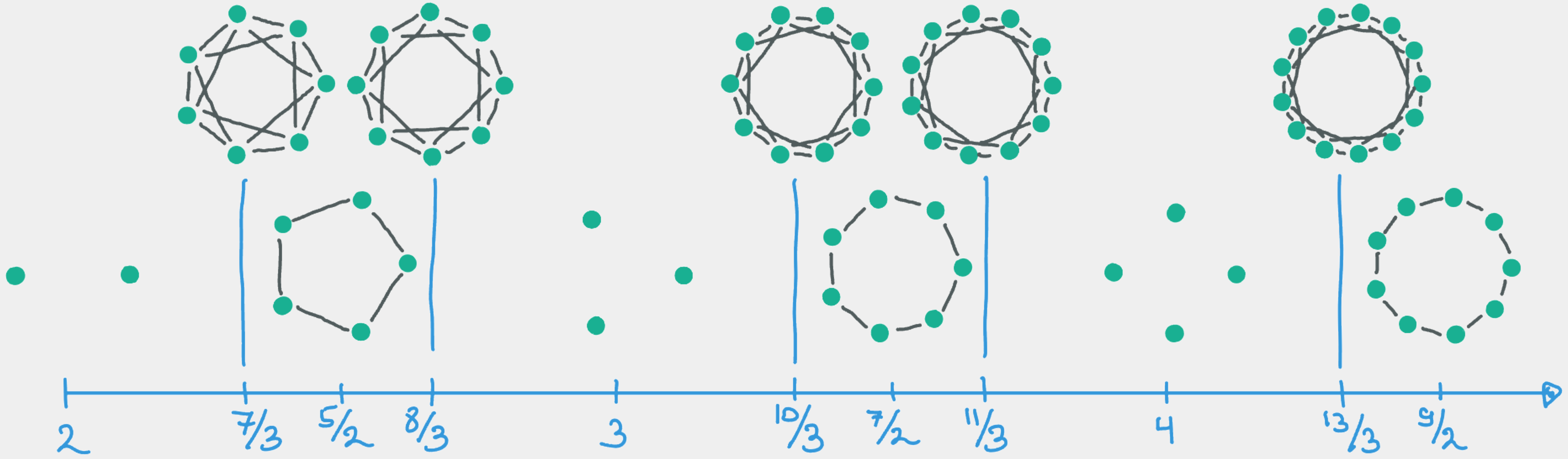


$E_{8/3}$

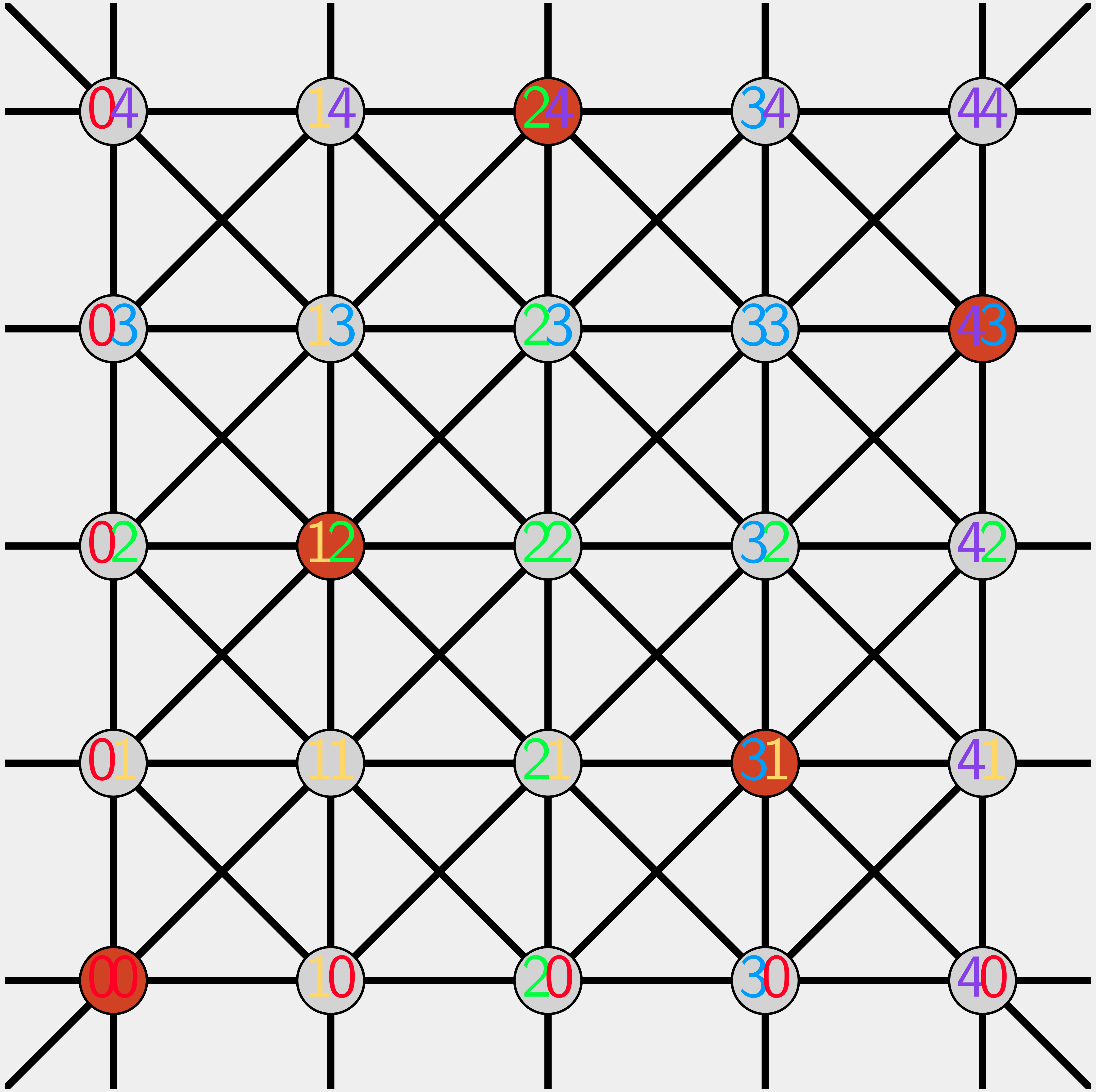
Theorem (Hell, Nešetřil 2004): We have $E_{p/q} \leq E_{p'/q'}$ iff $p/q \leq p'/q'$.

Theorem (De Boer, B., Zuiddam 2019):

- If p_n/q_n converges to p/q from above then E_{p_n/q_n} converges to $E_{p/q}$.
- If p_n/q_n converges to irrational r then E_{p_n/q_n} is Cauchy but has no limit in $\mathcal{G}$.



The independent set that yields $\alpha(C_5^2) = 5$ is actually a subgroup of $\mathbb{Z}_5^2$.



The independent set that yields $\alpha(C_5^2) = 5$ is actually a subgroup of $\mathbb{Z}_5^2$.

Every current best lower bound of the Shannon Capacity of odd cycles is derived from a subgroup in some nearby $E_{p/q}$.

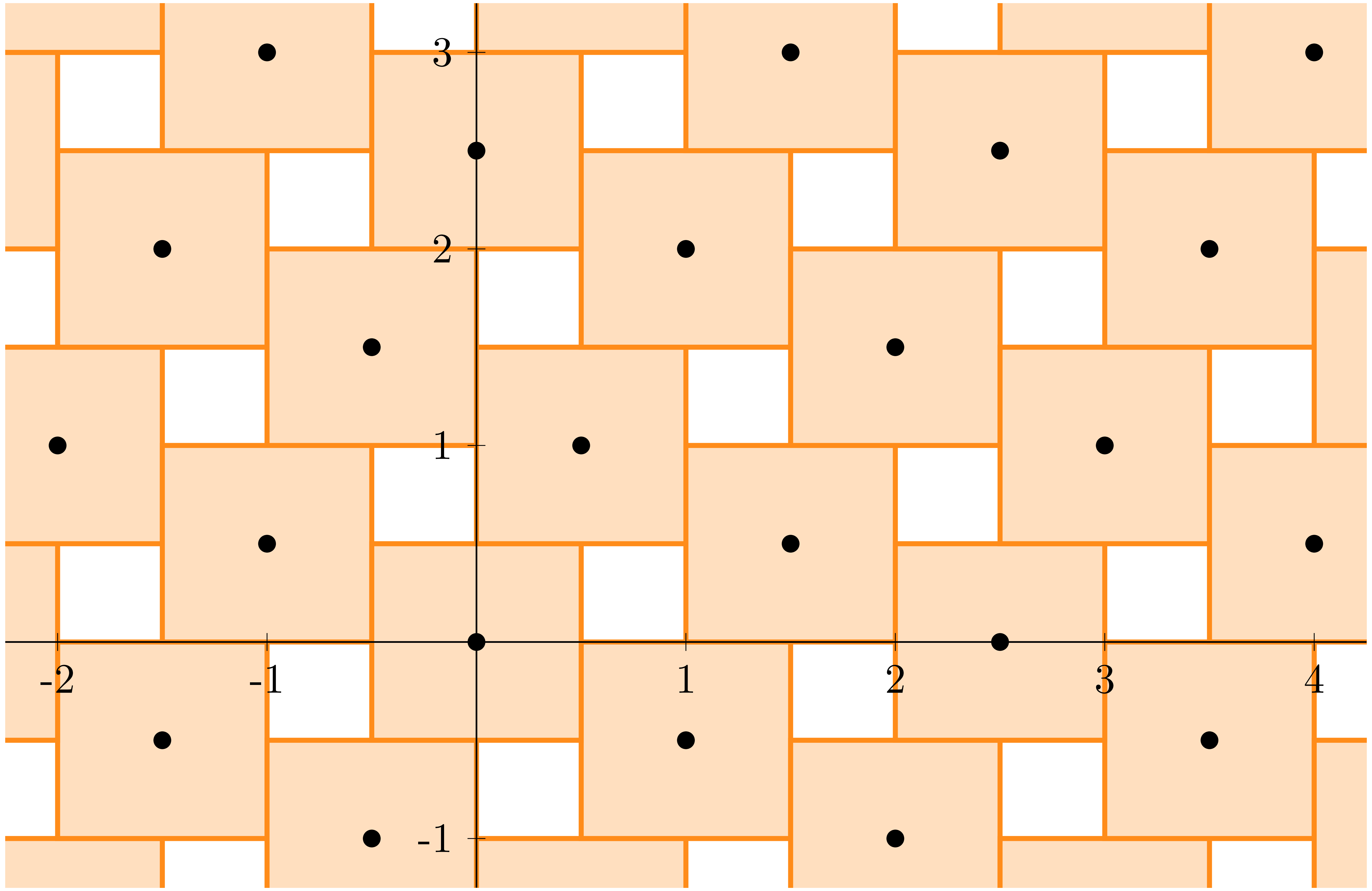
G	H	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1,7,7^2,7^3,7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1,0,2) + t \cdot (0,1,4) : s,t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR+71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1,11,11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR+71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1,19,117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR+71][18]
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1,15,1073,1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

Definition

For an (invertible) matrix $A \in \mathbb{R}^{n \times n}$, let $\mathscr{L}(A)$ denote the lattice spanned by its columns.

$$\mathscr{L}(A) = \{Av : v \in \mathbb{Z}^n\}$$

For a lattice Λ we let $\lambda_\infty(\Lambda) = \min\{\, |v|_\infty : v \in \Lambda,\ v \neq 0\}$.

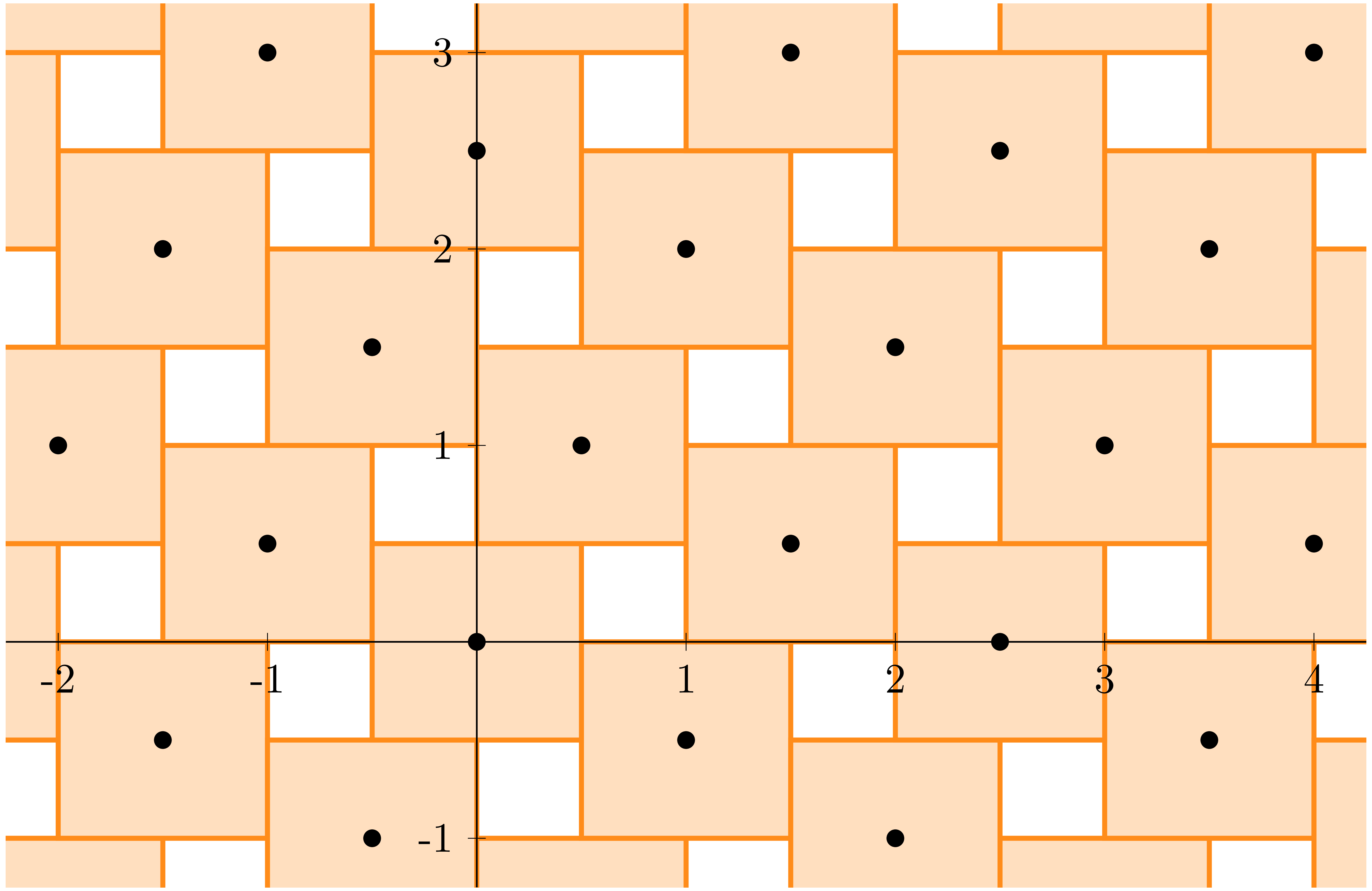


$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$\lambda_\infty(\mathscr{L}(A)) = 1$$

Lemma

Let $p/q \in \mathbb{Q}_{\geq 2}$, $A \in \mathbb{Q}^{n\times n}$ and $B \in \mathbb{Z}^{n\times n}$ such that



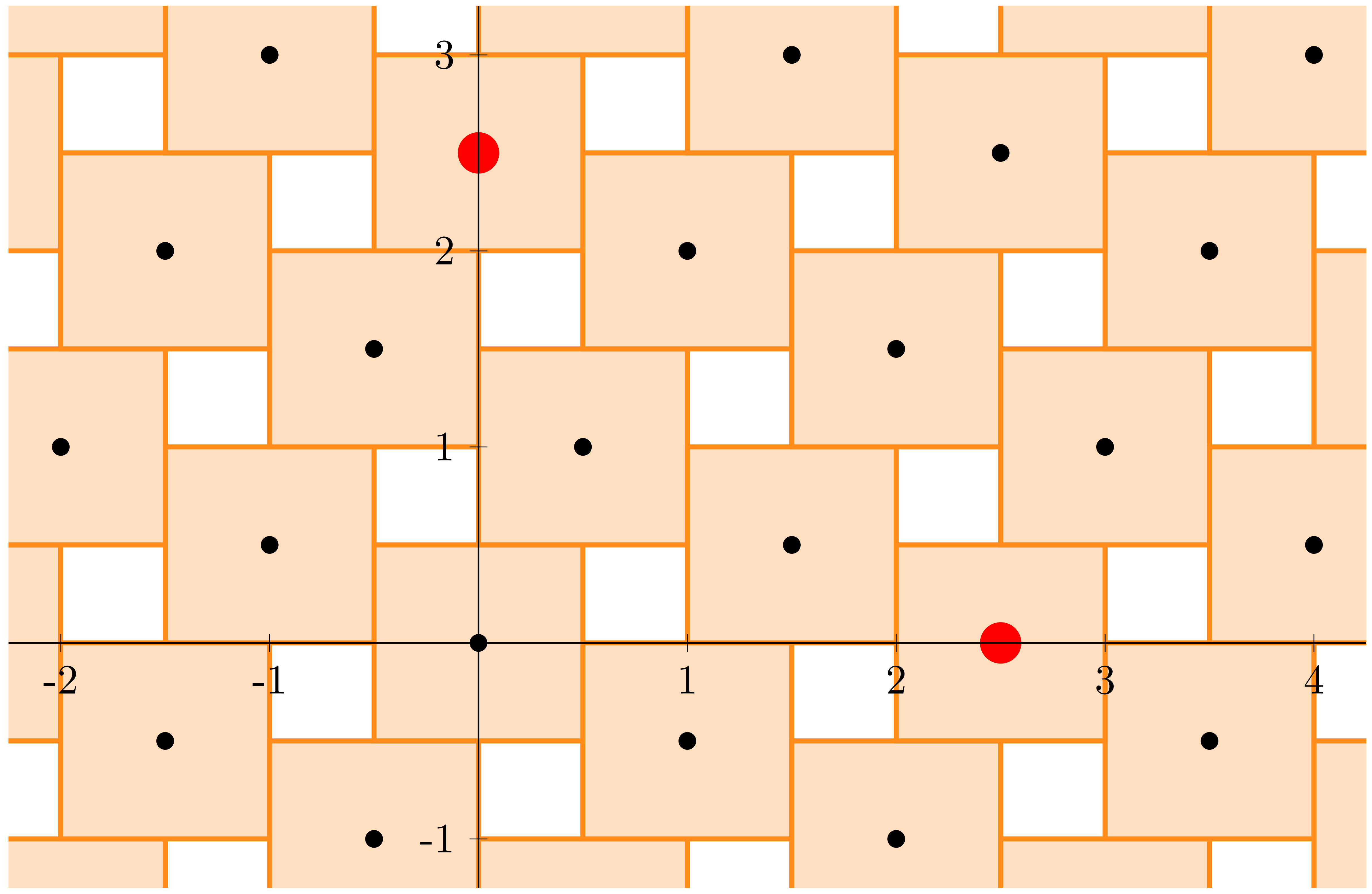
$\mathscr{L}(A)$

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad r = 5/2$$

Lemma

Let $p/q \in \mathbb{Q}_{\geq 2}$, $A \in \mathbb{Q}^{n\times n}$ and $B \in \mathbb{Z}^{n\times n}$ such that

- $AB = r\, I_n$



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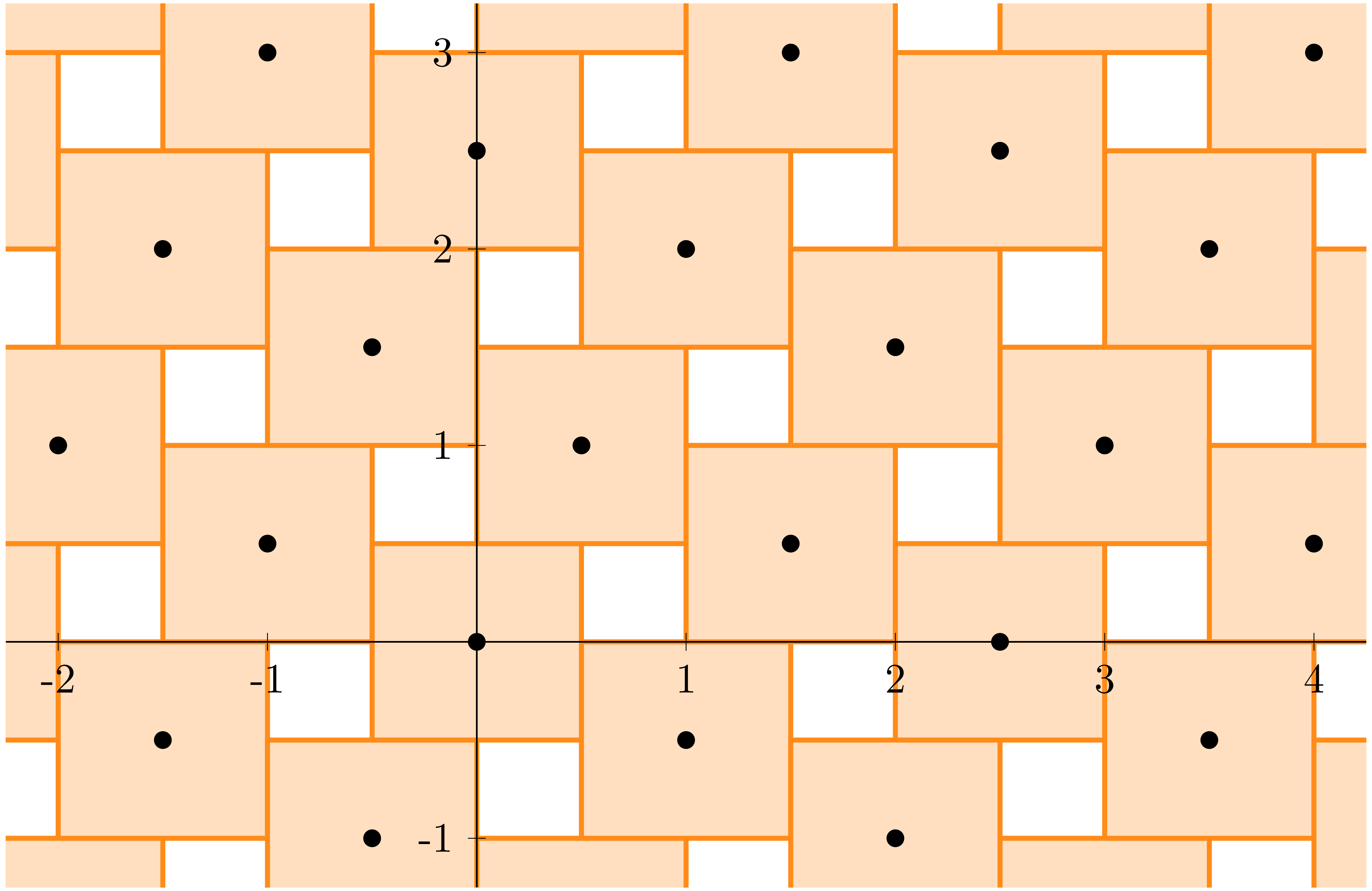
- $\lambda_\infty(\mathcal{L}(A)) \geq 1$



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Then $\alpha_n(r) \geq |\det(B)|$.

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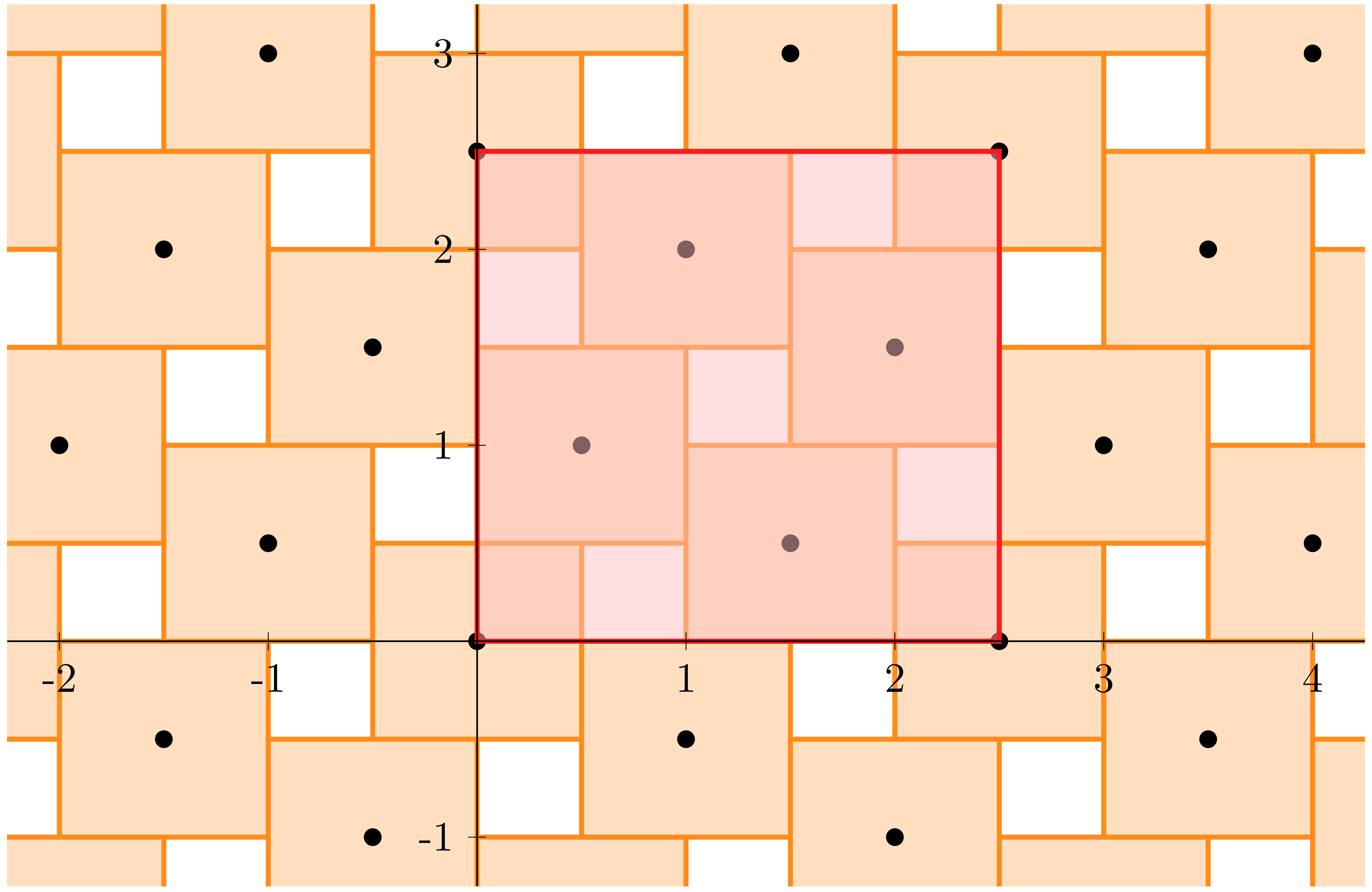
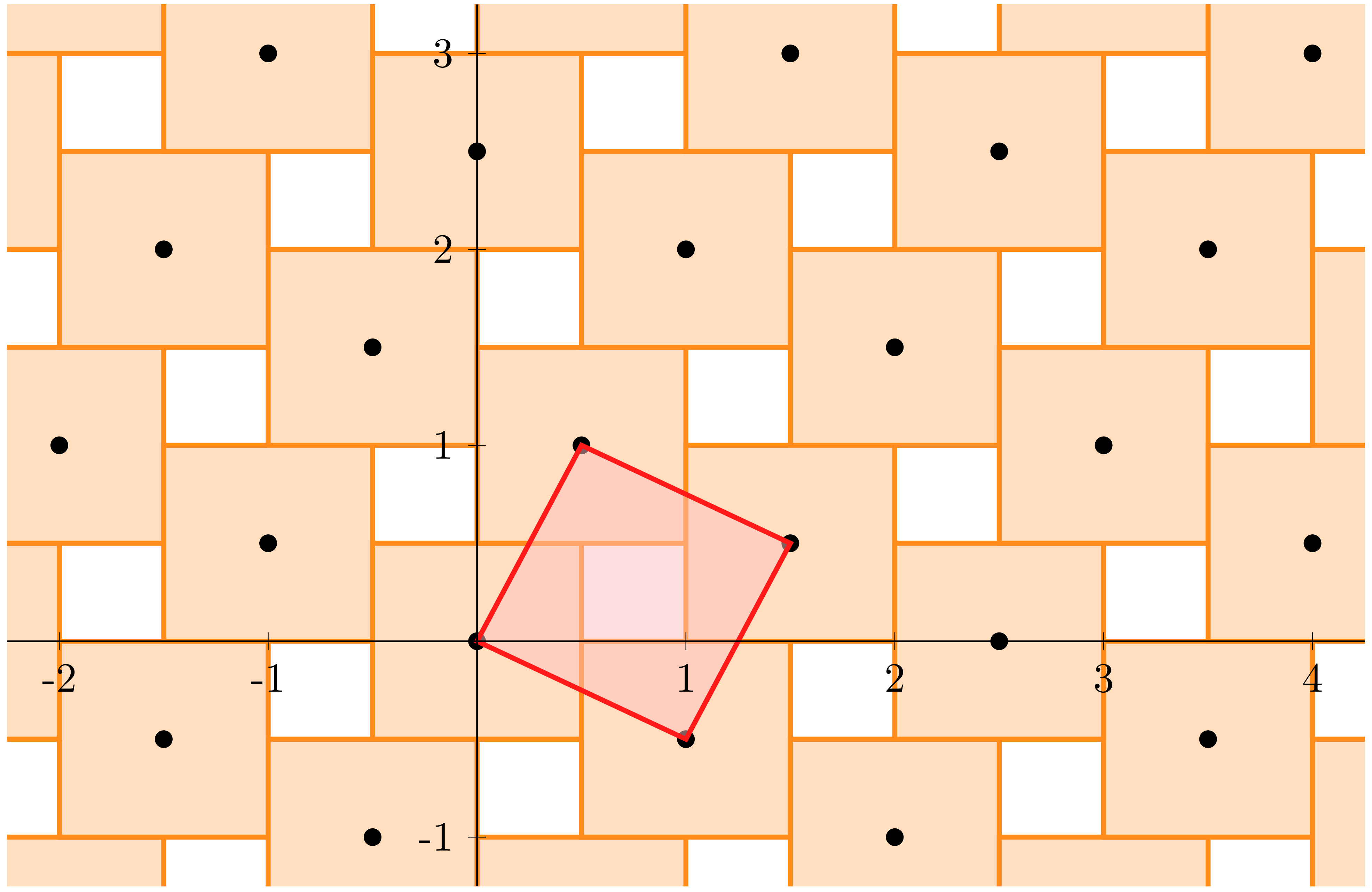


Volume torus is $r^n = \det(rI_n)$



Volume fundamental domain is $|\det(A)|$

Lemma

Let $r \in \mathbb{R}_{\geq 1}$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{Z}^{n\times n}$ such that

- $AB = r\, I_n$
- $\lambda_\infty(\mathscr{L}(A)) \geq 1$

Then $\alpha_n(r) \geq |\det(B)|$.

Example (B., Polak, Zuiddam; 2025)

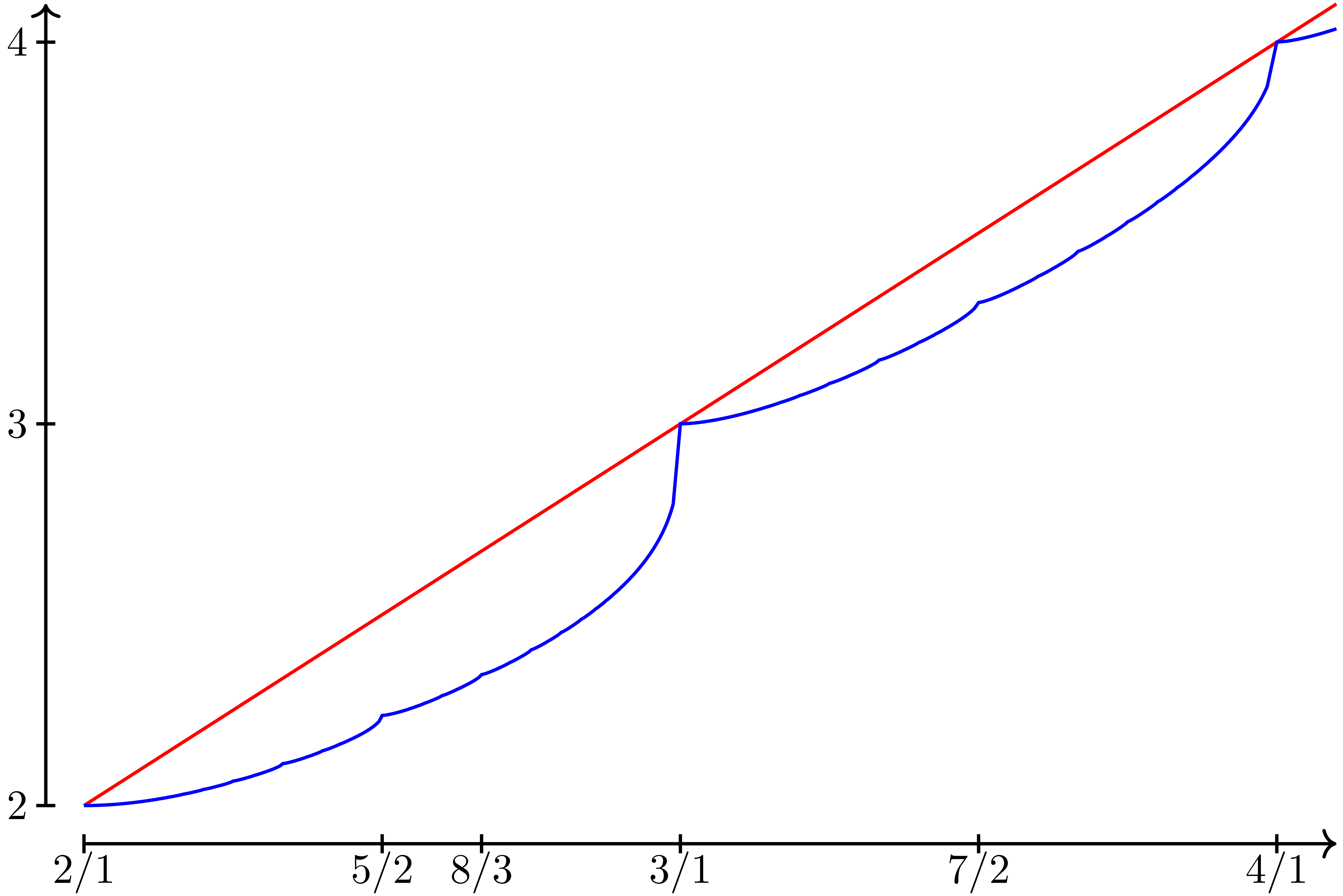
For $n = 5$ and parameters k, b, r, s let

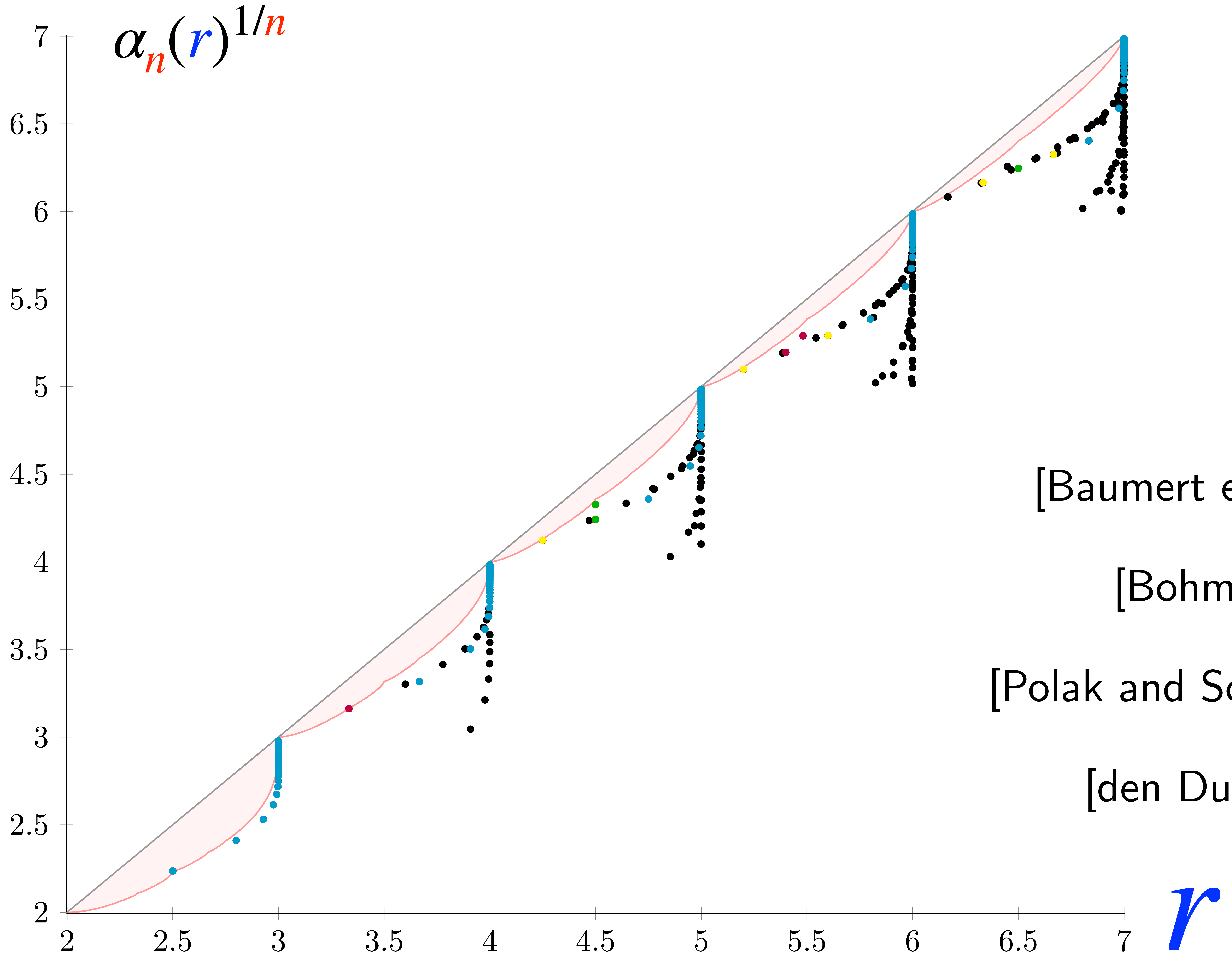
$$B = \begin{pmatrix} b^4k+s & -r & -br & -b^2r & -b^3r \\ b^3k & b^4k+s & -r & -br & -b^2r \\ b^2k & b^3k & b^4k+s & -r & -br \\ bk & b^2k & b^3k & b^4k+s & -r \\ k & bk & b^2k & b^3k & b^4k+s \end{pmatrix} \qquad r = \frac{k(b^5k+bs+r)^5+rs^5}{bk(b^5k+bs+r)^4+rs^4}$$

and $A = rB^{-1}$. These satisfy the assumptions of the lemma (under some constraints).

Theorem [Bachoc, Pêcher and Thiéry; 2013]

$$\Theta\left(E_{p/q}\right) \leq \frac{p}{q} \sum_{i=0}^{q-1} \prod_{j=1}^{q-1} \frac{\cos(\frac{2i\pi}{q}) - \cos(\lfloor \frac{pj}{q} \rfloor \frac{2\pi}{p})}{1 - \cos(\lfloor \frac{pj}{q} \rfloor \frac{2\pi}{p})} = \vartheta(E_{p/q})$$



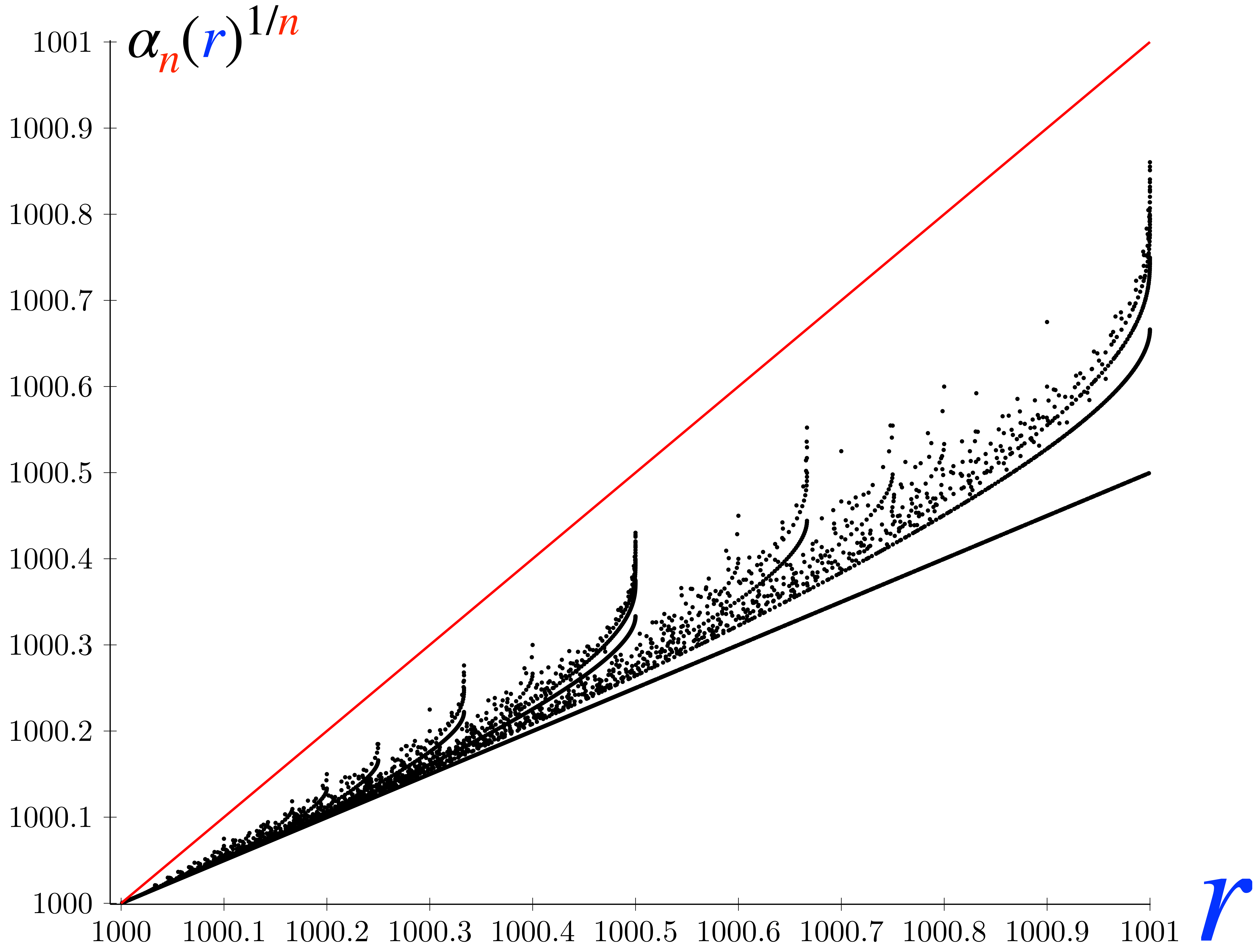


[Baumert et al.; 1971]

[Bohman; 2003]

[Polak and Schrijver; 2019]

[den Dulk; 2022]



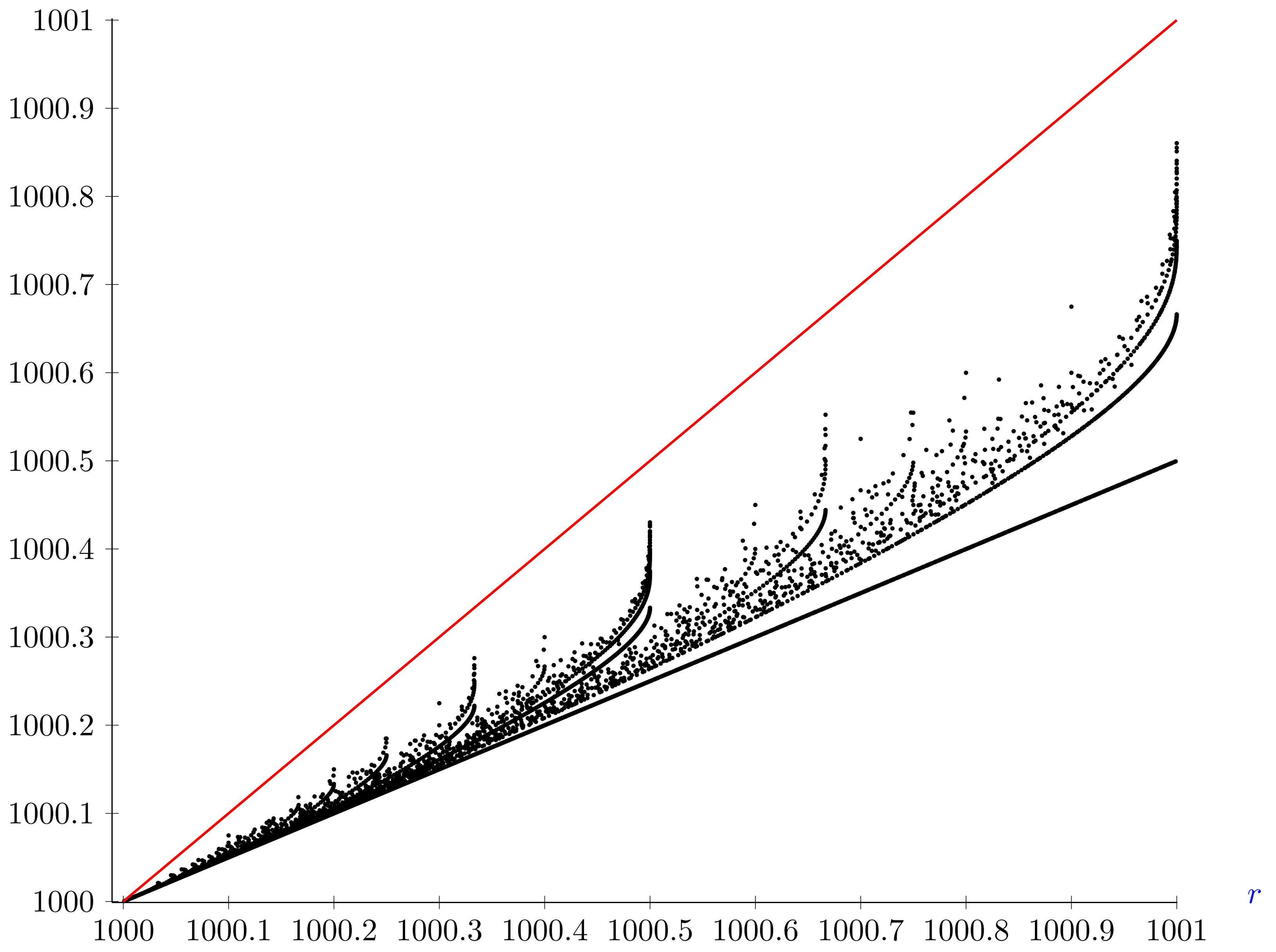
$\alpha_n(r)^{1/n}$
1001
1000.9
1000.8
1000.7
1000.6
1000.5
1000.4
1000.3
1000.2
1000.1
1000
1000
1000.1
1000.2
1000.3
1000.4
1000.5
1000.6
1000.7
1000.8
1000.9
1001
r

Theorem [Bohman; 2003]

We have $\lim_{m\to\infty} (m+1/2) - \tilde{\Theta}(m+1/2) = 0$ (with $m \in \mathbb{N}$.)

Theorem [B., Polak, Zuiddam; 2025]

We have $\lim_{r\to\infty} r - \tilde{\Theta}(r) = 0.$



Theorem [B., Polak, Zuiddam; 2025]

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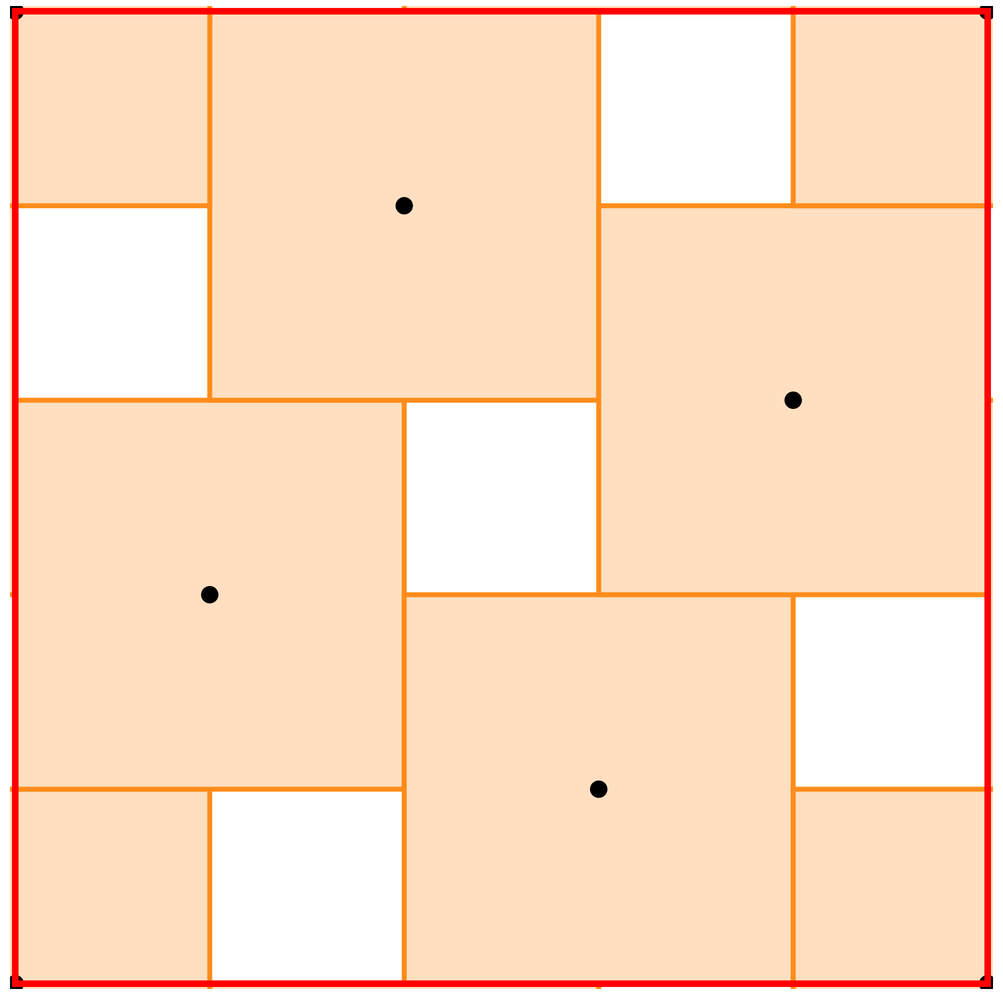
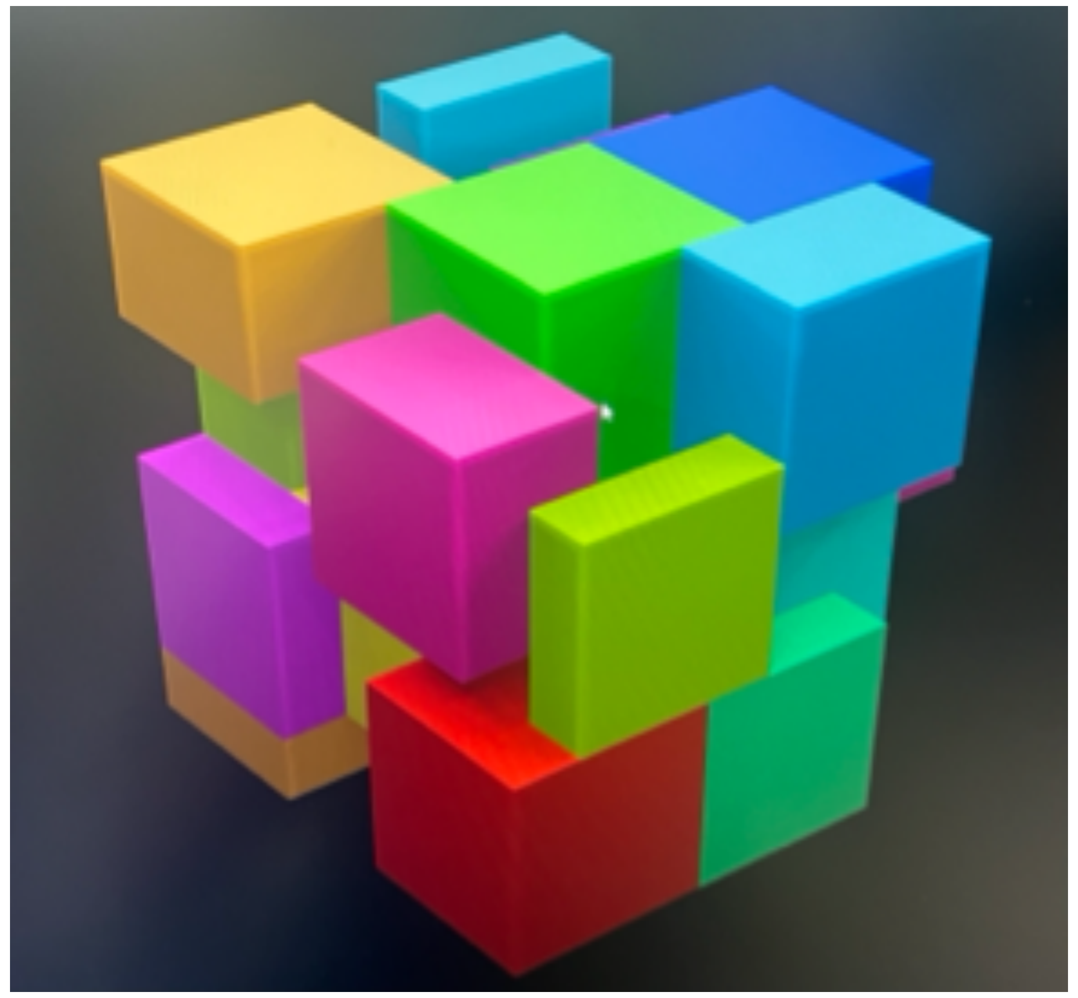


$r = 5/2$



$r = 8/3$

Open Problem

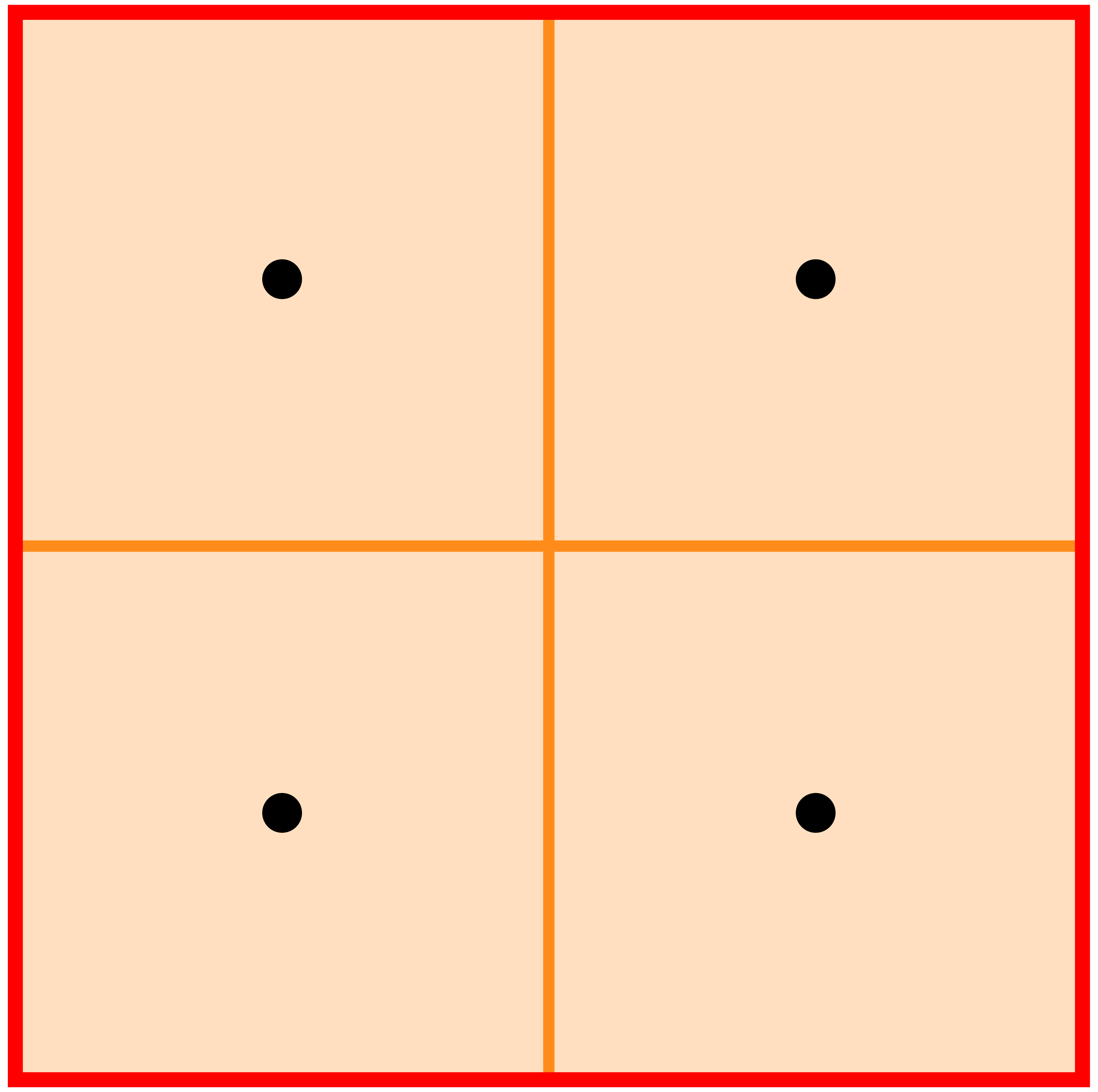
Let $\Theta_{\mathrm{grp}}(r)$ denote the asymptotic packing density of side length r torus with lattices.

Do we have $\Theta_{\mathrm{grp}}(r) = \tilde{\Theta}(r)$?

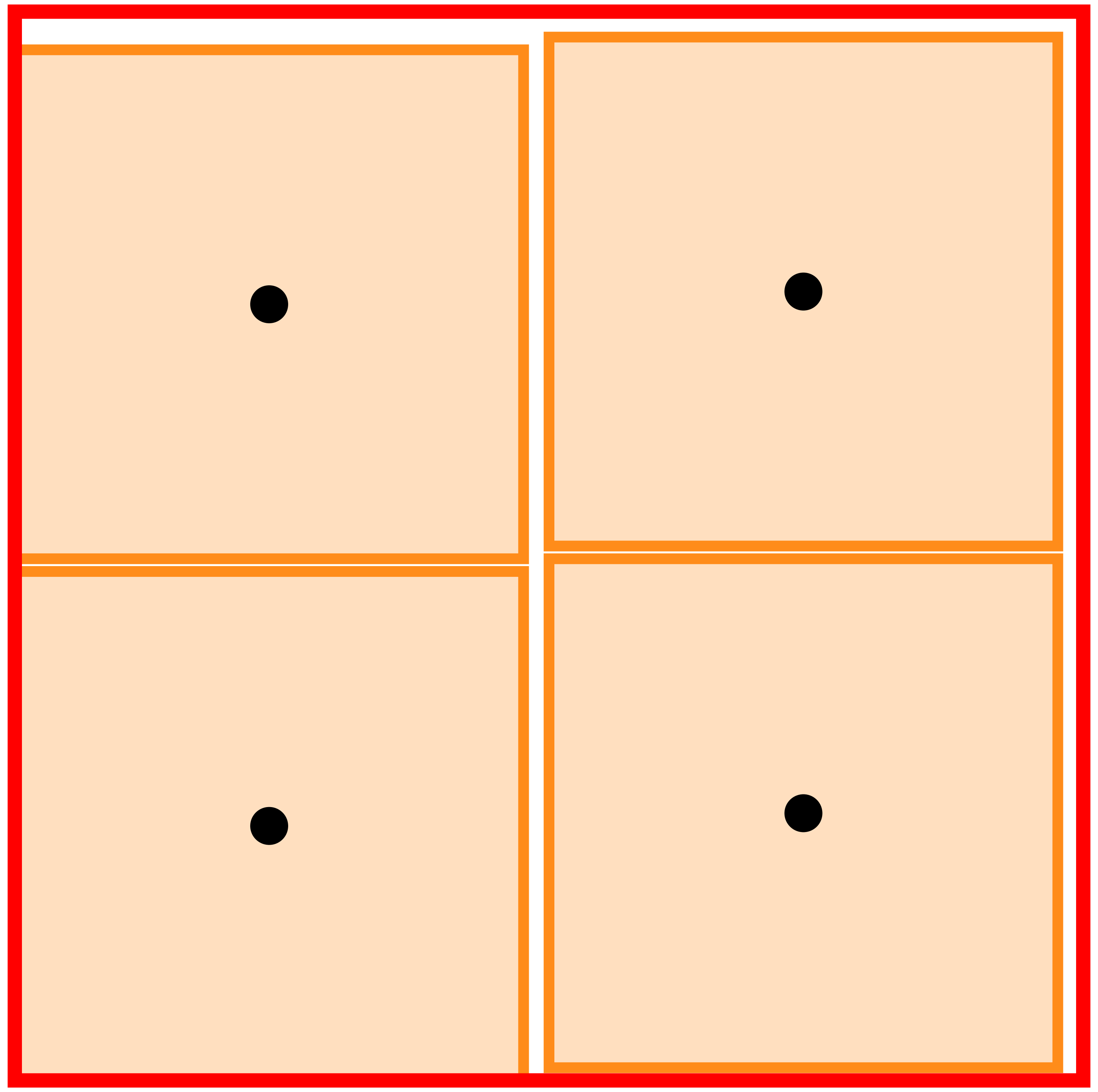
Theorem [Bohman and Holzman; 2003]

Whenever $r > 2$ then $\tilde{\Theta}(r) > 2$.

$r = 2$



$r > 2$



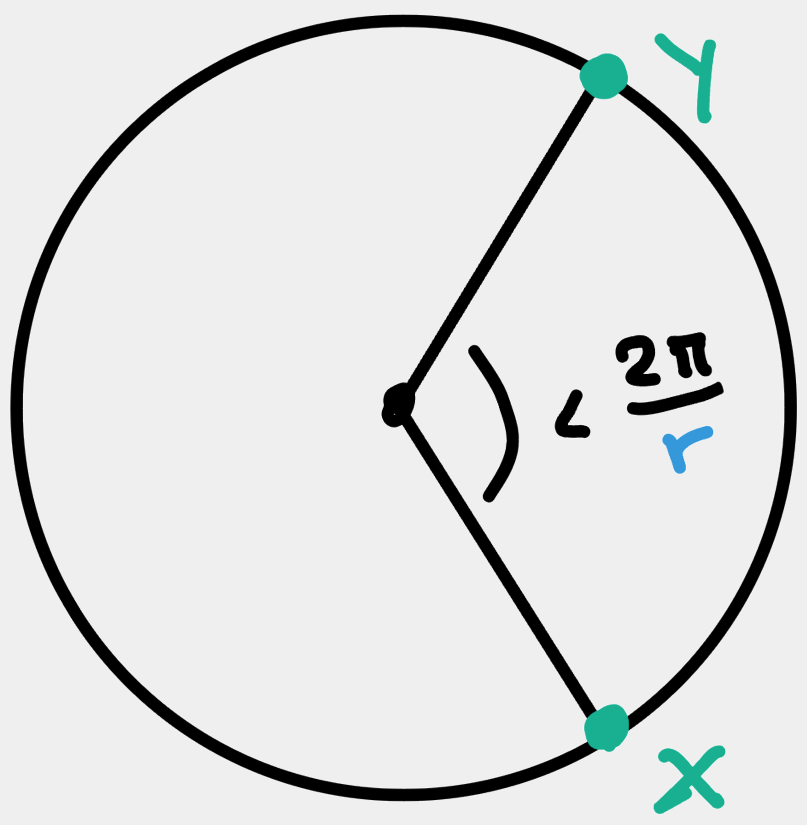
Q. Is Θ strictly increasing? I.e. if $r_1 < r_2$ then $\Theta(r_1) < \Theta(r_2)$?

Q. What about Θ_{grp}?

Q. Is there a concrete and intuitive description of the completion of $\mathscr{G}$?

If p_n/q_n converges to an irrational number r then E_{p_n/q_n} is Cauchy.

The sequence does converge within the space of infinite graphs $\mathscr{G}_\infty$ to a graph on S^1:



Q. Does every Cauchy Sequence of finite graphs converge within $\mathscr{G}_\infty$?

Q. Is $\mathscr{G}_\infty$ complete?

Q. Is every (infinite) graph $G \in \mathscr{G}_\infty$ the limit of finite graphs? (We think no.)