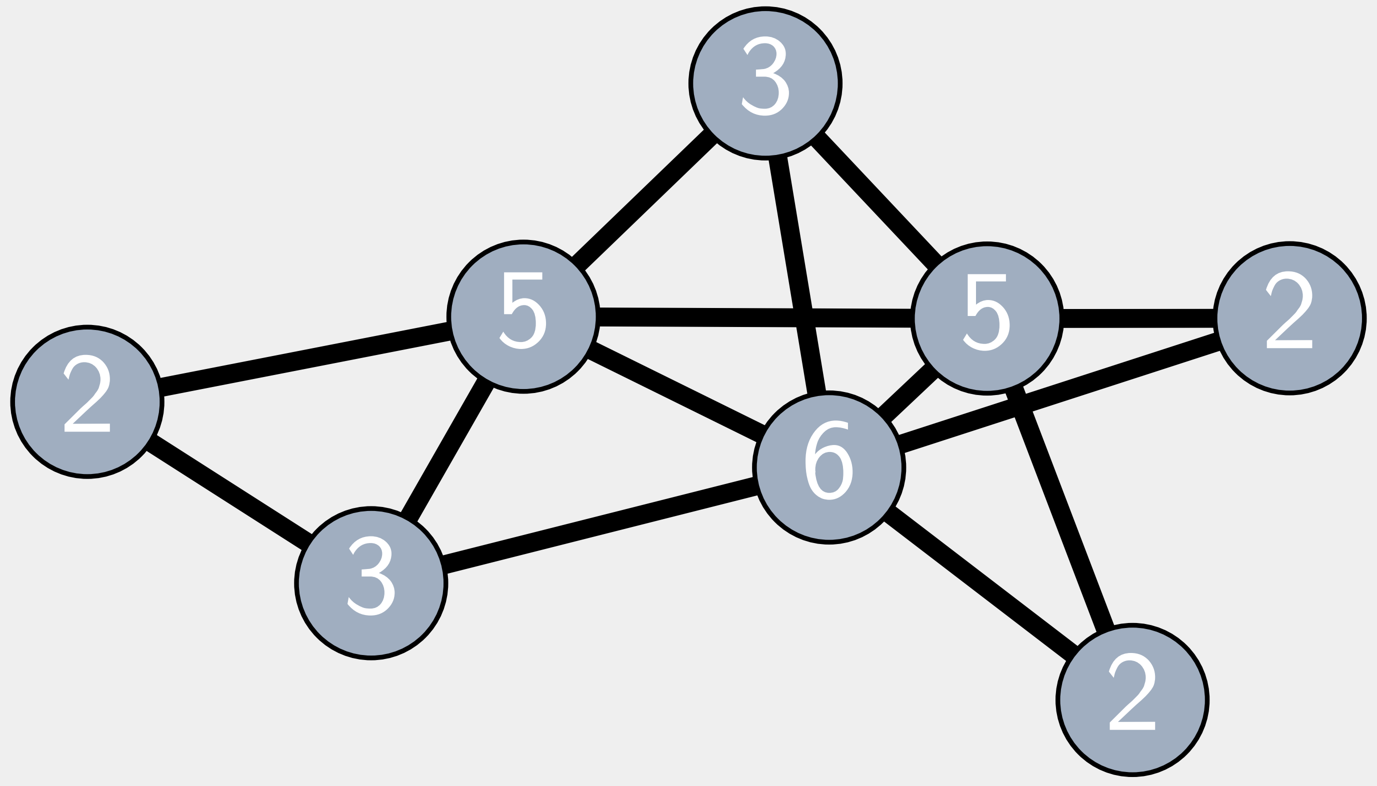
Triangle-free graphs with the fewest independent sets

Pjotr Buys[1], Jan van den Heuvel[2], and Ross Kang[1]

1. University of Amsterdam
2. London School of Economics

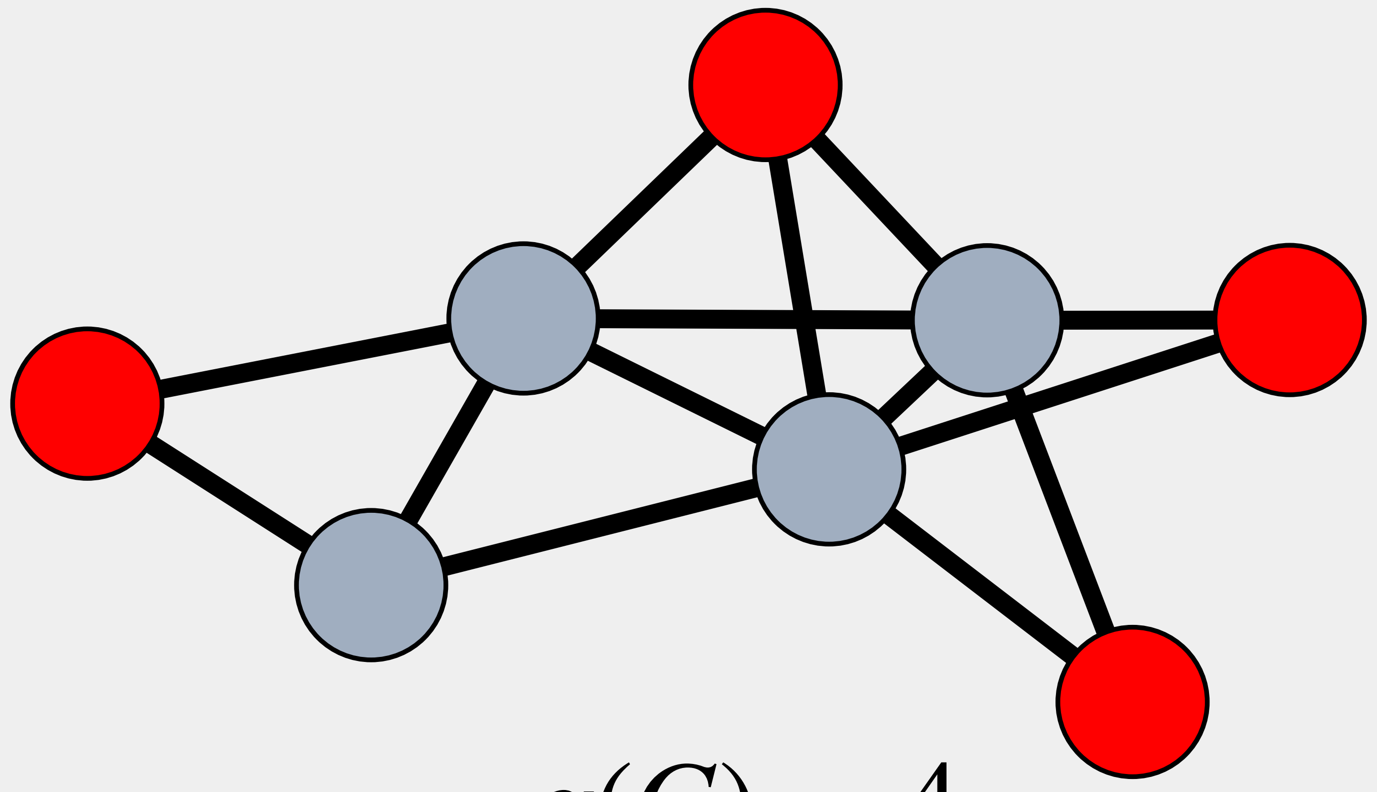
Given a graph G the average degree of G is

$$d(G) := \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg_G(v)$$



$$\frac{1}{8}\left(2+3+5+3+6+5+2+2\right) = \frac{7}{2}$$

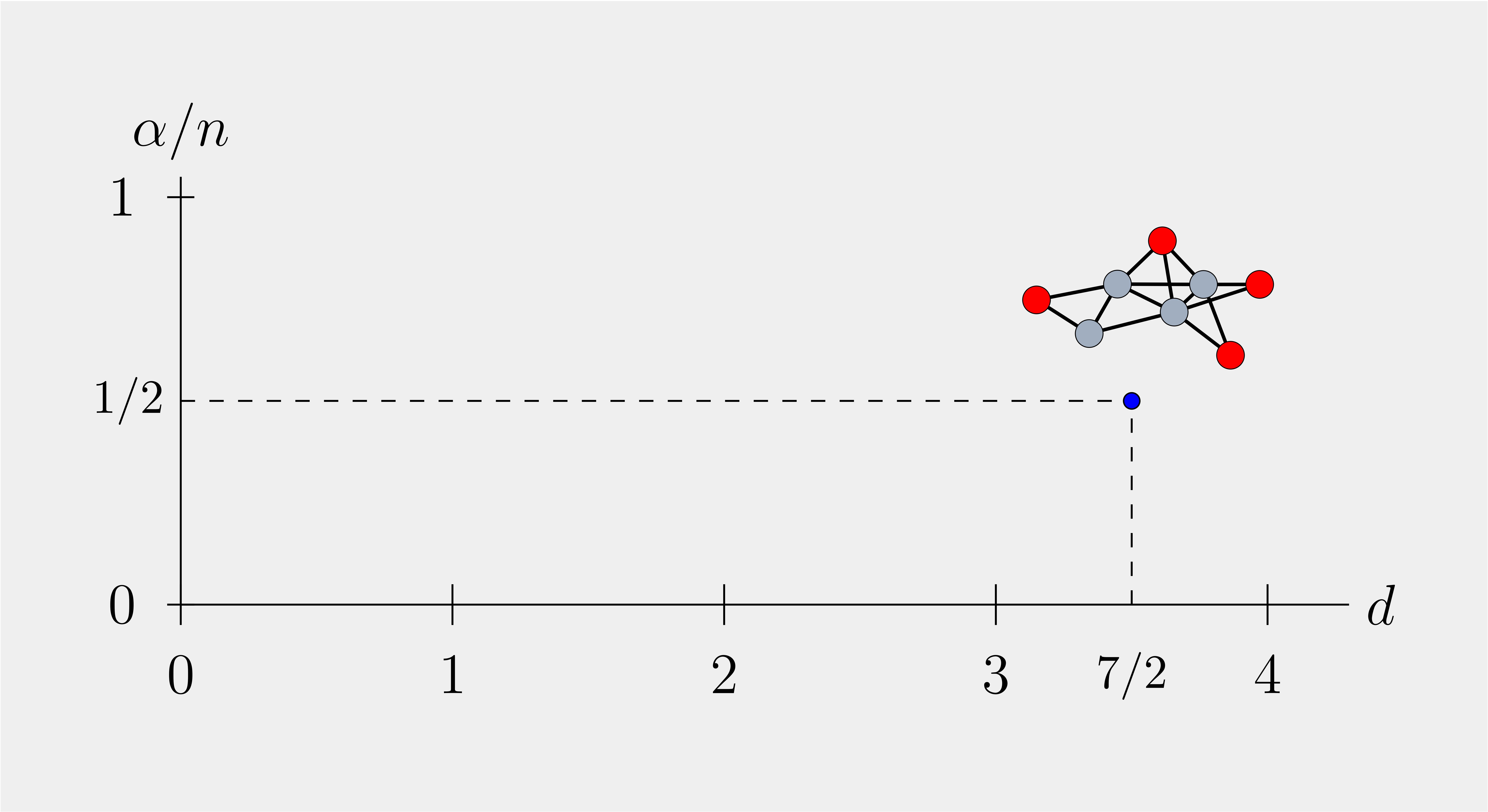
A subset $I \subseteq V(G)$ is called an independent set if it does not span any edges.



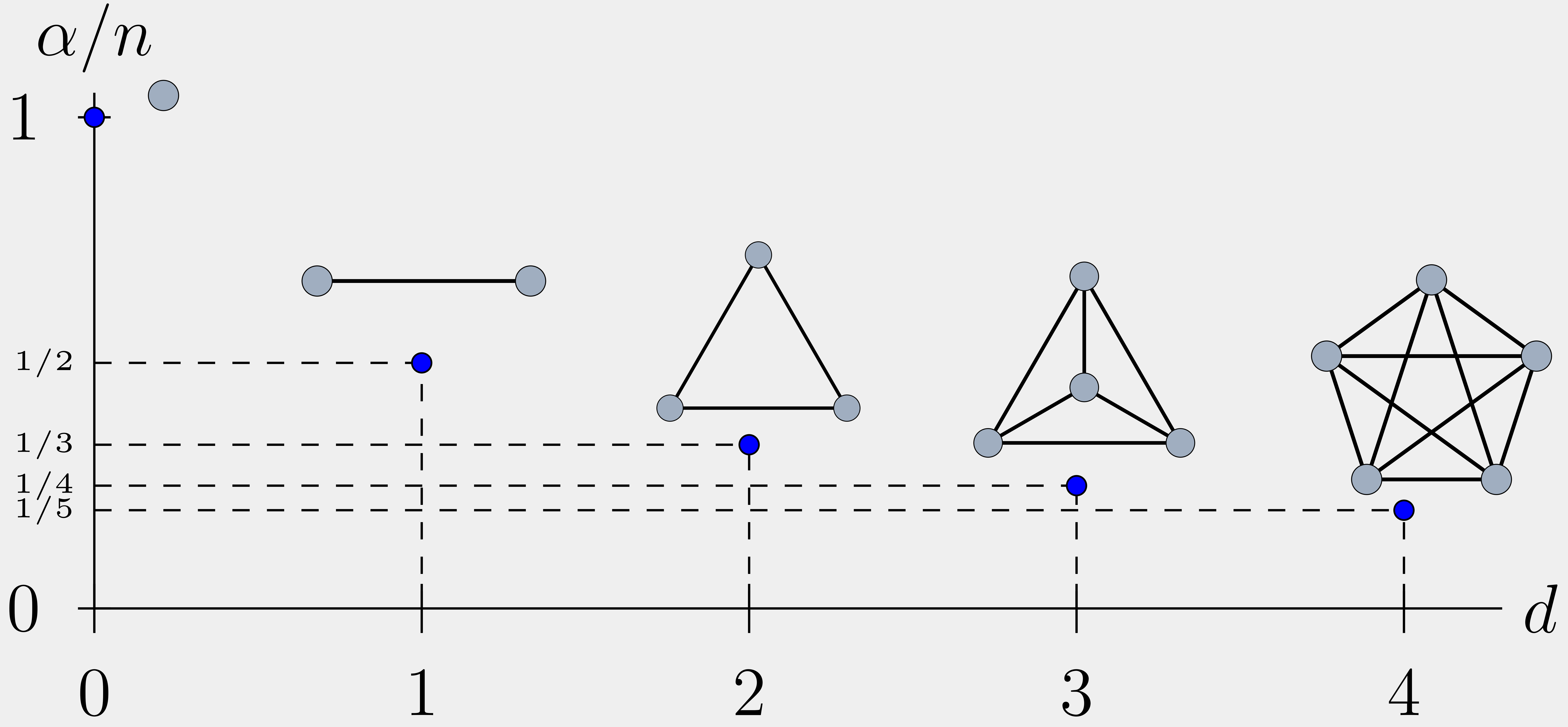
$\alpha(G) = 4$

The maximum size of an independent set is $\alpha(G)$.

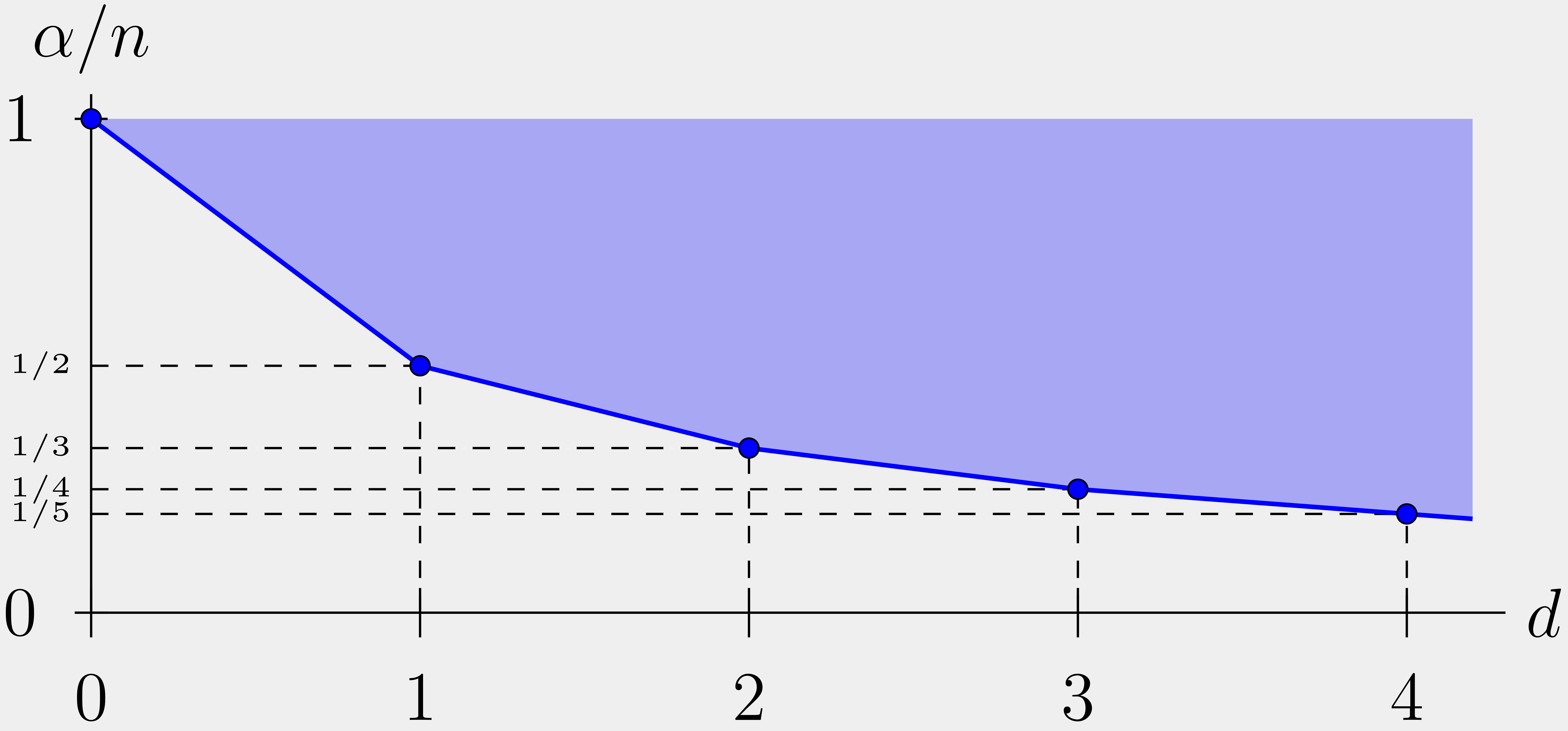

α/n
1
0
0
1
2
3
4
d



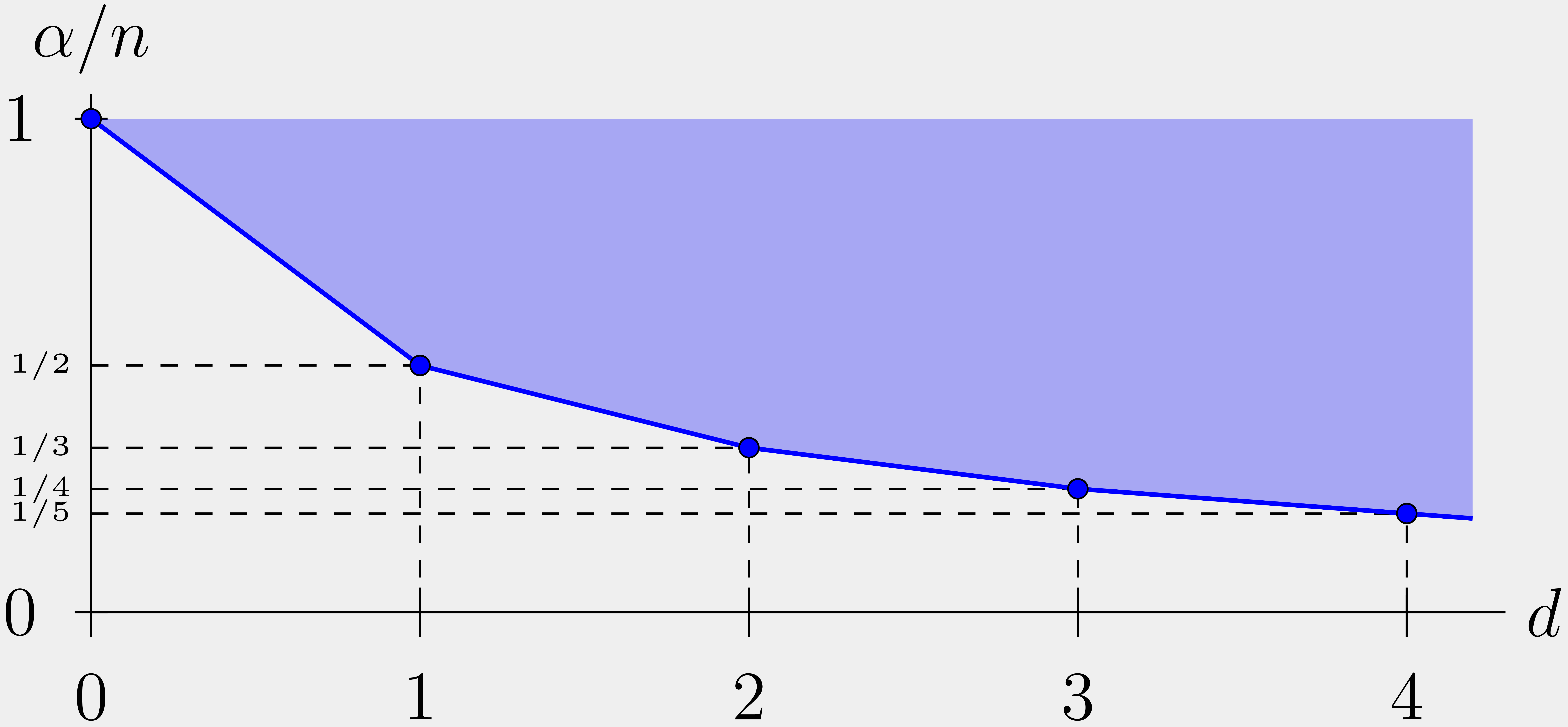
α/n
1
1/2
0
d
0
1
2
3
7/2
4



α/n
1
1/2
1/3
1/4
1/5
0
d
0
1
2
3
4

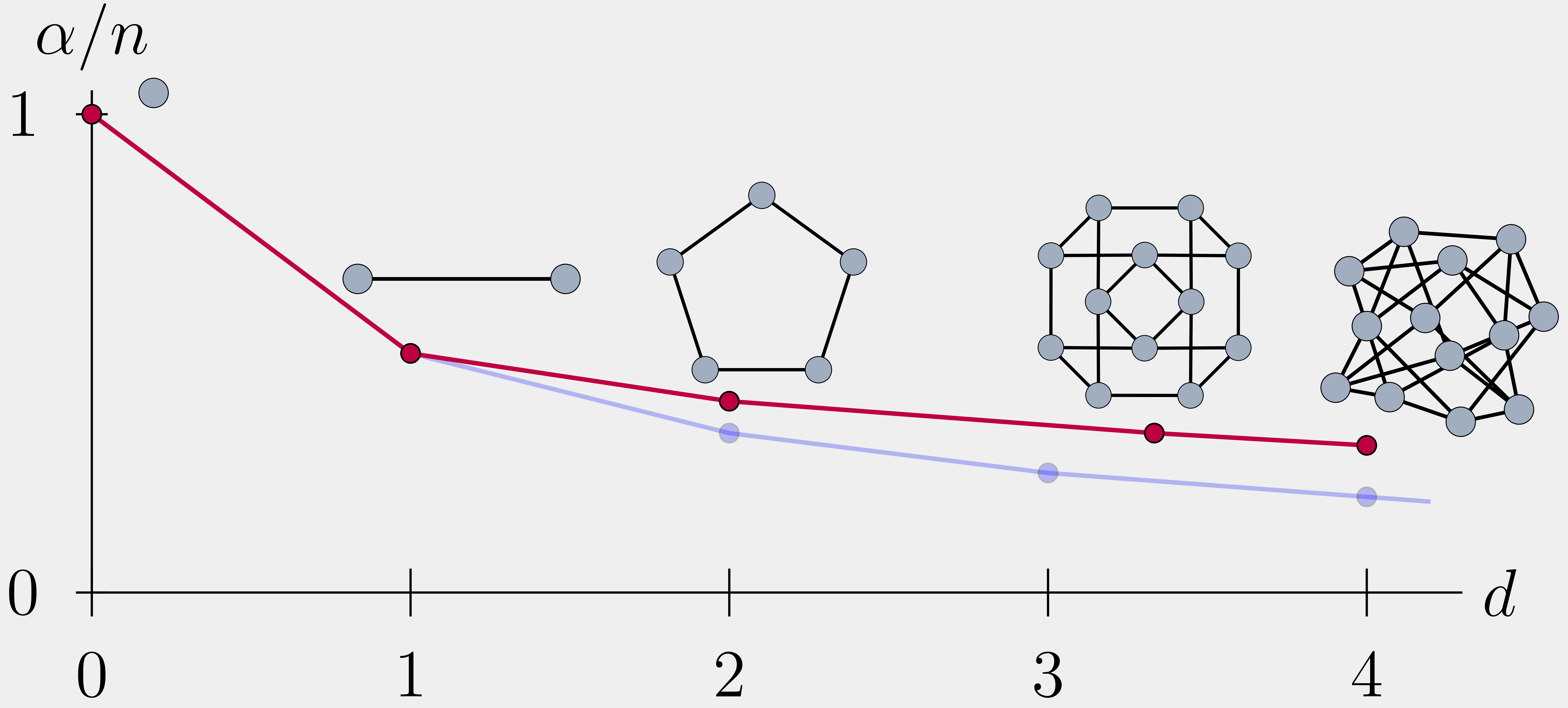


Turán's Theorem $\implies$ Every graph lies in the blue region.



Define $\mathscr{G}(d) = \inf\left\{\alpha(G)/n(G)\,|\,G \text{ average degree at most d}\right\}$ then $\mathscr{G}(d) \geq \dfrac{1}{d+1}$.

Triangle-Free Graphs



Define

$$\mathcal{G}(d) = \inf \left\{ \alpha(G)/n(G) \,\middle|\, G \text{ triangle-free and average degree at most d} \right\}.$$

Theorem (Ajtai, Komlós, Szemerédi; 1980):

$$0.01 \cdot \frac{\log d}{d} \leq \mathcal{G}(d)$$

Theorem (Shearer; 1983):

$$\left(1 \cdot \frac{\log d}{d} \sim \right) \qquad \frac{d \log(d) - d + 1}{(d-1)^2} \leq \mathcal{G}(d)$$

The random graph $G\left(n, d/n\right)$ shows that asymptotically

$$\mathcal{G}(d) \leq 2 \cdot \frac{\log d}{d}$$

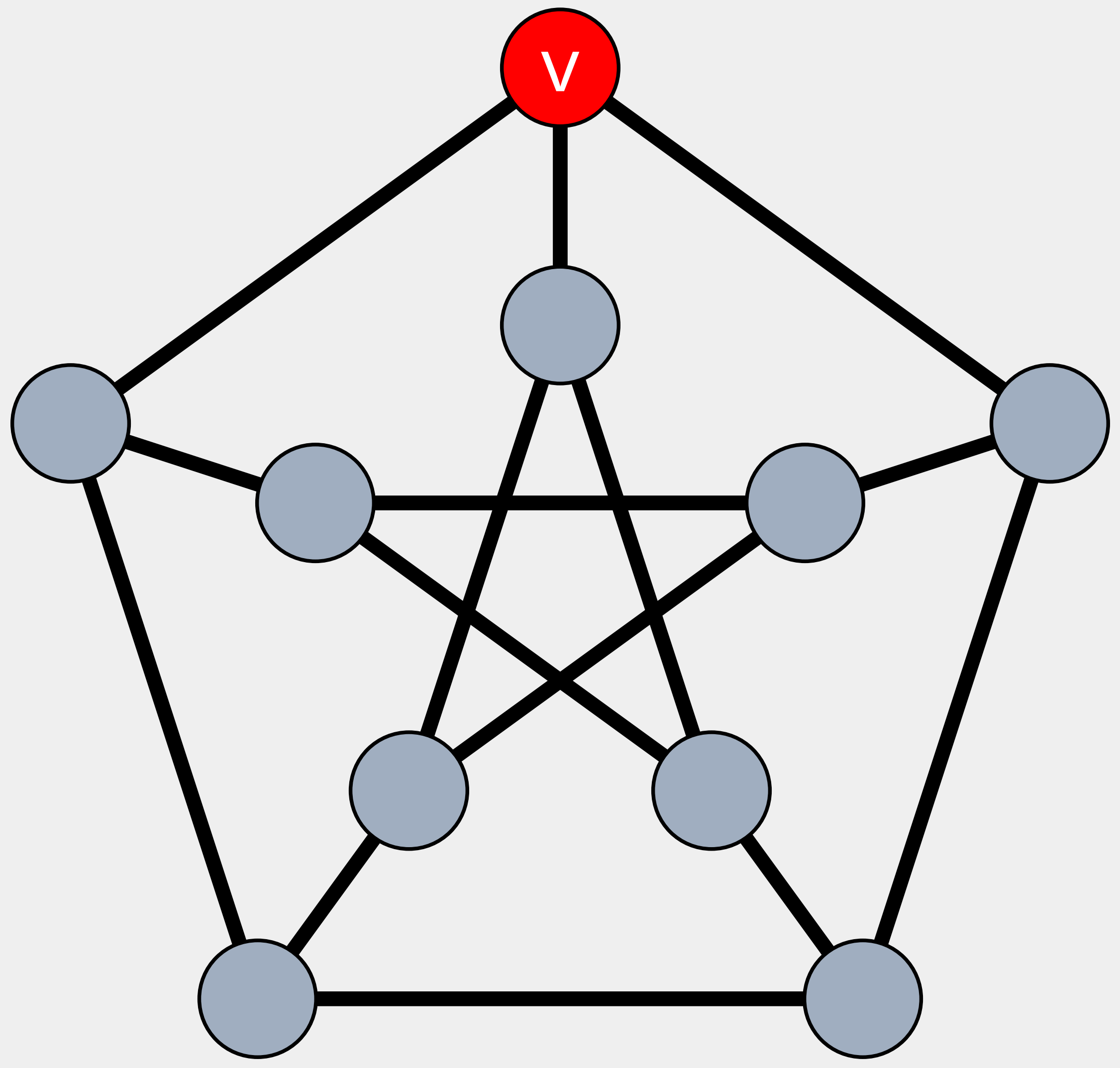
Theorem (Shearer; 1983):

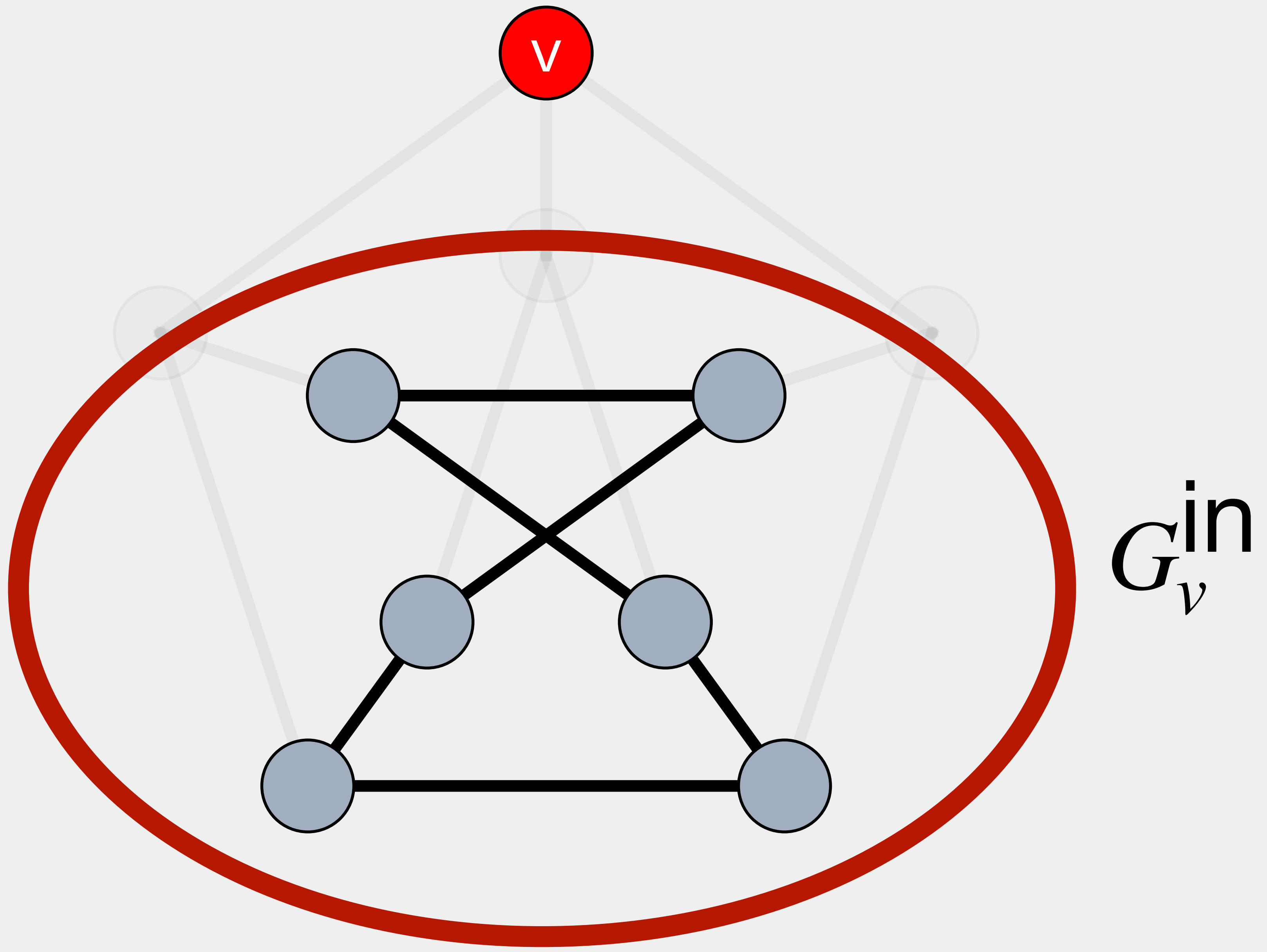
Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$\alpha(G) \geq f(d) \cdot |V(G)|.$$

Proof. Take any $v \in V(G)$, then:

$$\alpha(G) \geq 1 + \alpha(G_v^{\text{in}})$$





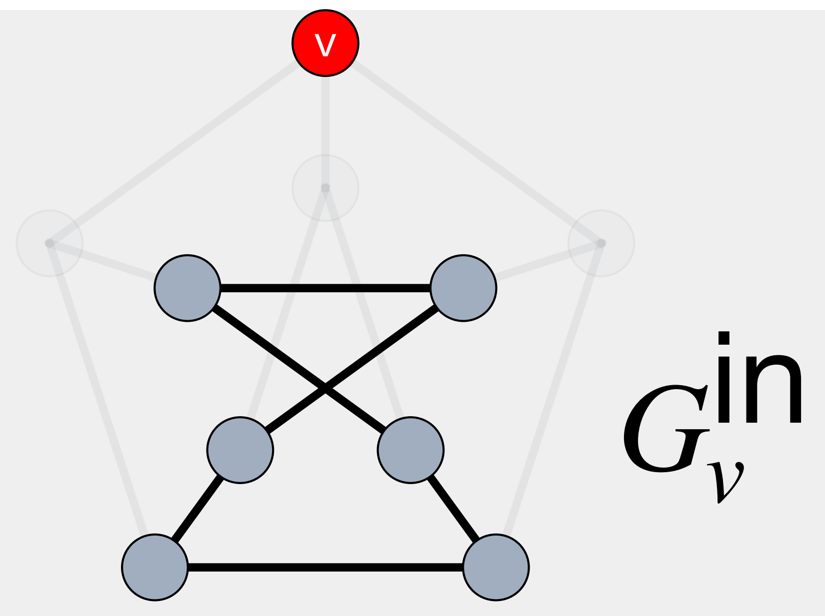
i) Differentiable. ii) Decreasing. iii) Convex. iv)

Theorem (Shearer; 1983):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$\alpha(G) \geq f(d) \cdot |V(G)|.$$

Proof. Take any $v \in V(G)$, then:



$$\begin{aligned}\alpha(G) &\geq 1 + \alpha(G_v^{\text{in}}) \\ &\geq 1 + f(d(G_v^{\text{in}})) \cdot |V(G_v^{\text{in}})| \\ &\geq 1 + \left[f(d) + (d - d(G_v^{\text{in}}))f'(d)\right] \cdot |V(G_v^{in})|\end{aligned}$$

Pick v uniformly at random now the RHS in expectation is at least the following:

$$f(d) \cdot |V(G)| + 1 + (d - d^2)f'(d) - (1 + d)f(d)$$

Which is at least $f(d) \cdot |V(G)|$ by iv).

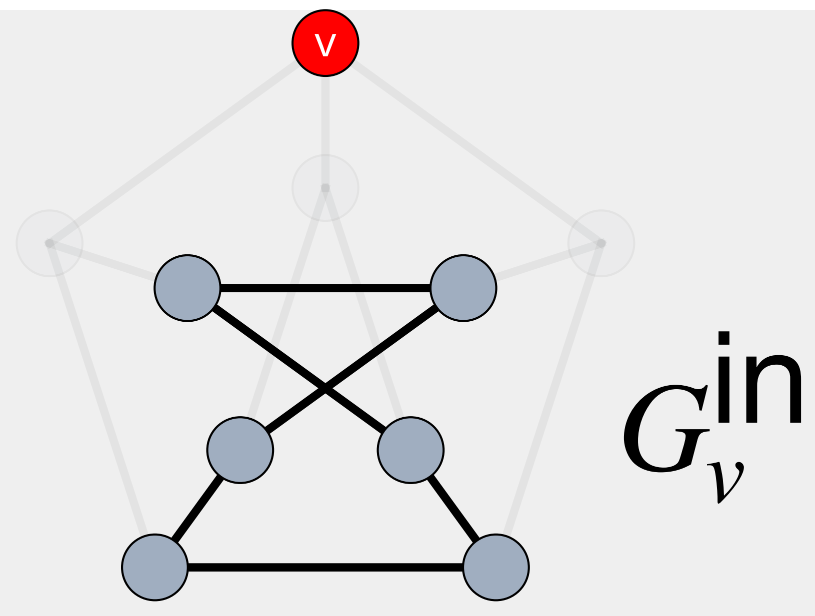
i) Differentiable. ii) Decreasing. iii) Convex. iv)

Theorem (Shearer; 1983):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$\alpha(G) \geq f(d) \cdot |V(G)|.$$

Proof. Take any $v \in V(G)$, then:



$$\alpha(G) \geq 1 + \alpha(G_v^{\text{in}})$$

$$\geq 1 + f(d(G_v^{\text{in}})) \cdot |V(G_v^{\text{in}})|$$

$$\geq 1 + \left[f(d) + (d - d(G_v^{\text{in}}))f'(d)\right] \cdot |V(G_v^{in})|$$

Pick v uniformly at random now the RHS in expectation is at least the following:

$$f(d) \cdot |V(G)| + 1 + (d - d^2)f'(d) - (1+d)f(d)$$

Which is at least $f(d) \cdot |V(G)|$ by iv).

i) Differentiable. ii) Decreasing. iii) Convex. iv) $1 + (x - x^2)f'(x) \geq (1+x)f(x)$

Theorem (Shearer; 1983):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$\alpha(G) \geq f(d) \cdot |V(G)|.$$

i) Differentiable. ii) Decreasing. iii) Convex. iv) $1 + (x - x^2)f'(x) \geq (1 + x)f(x)$

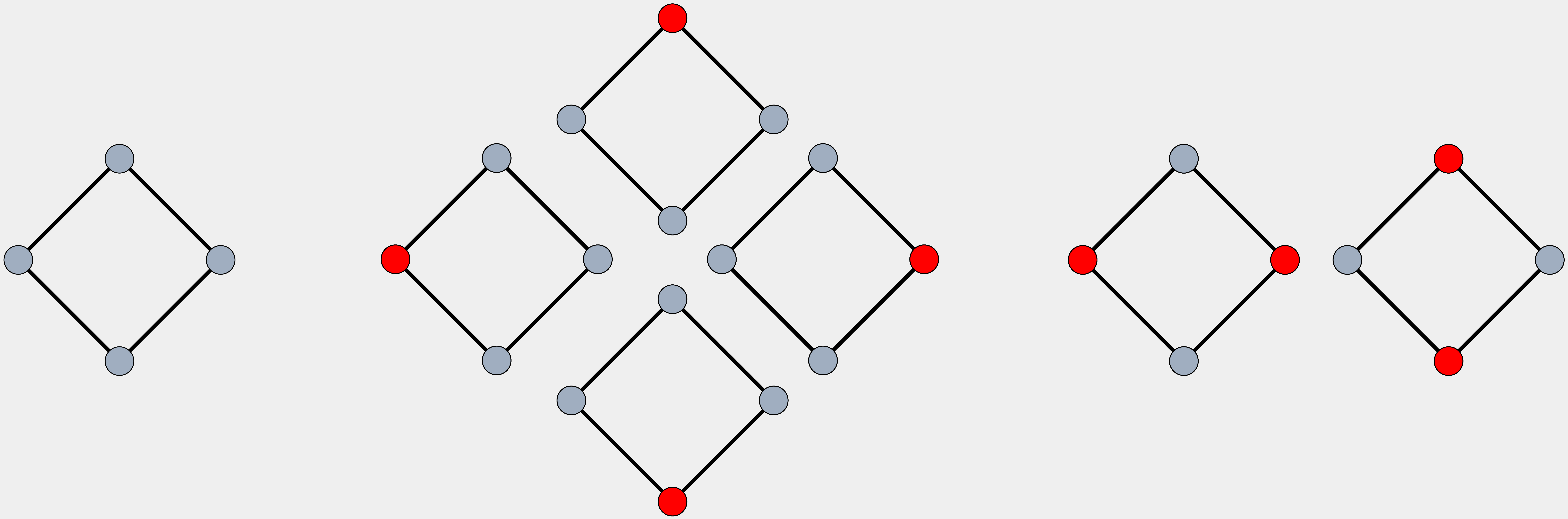
Theorem (Shearer; 1983):

The following function satisfies i), ii), iii), iv):

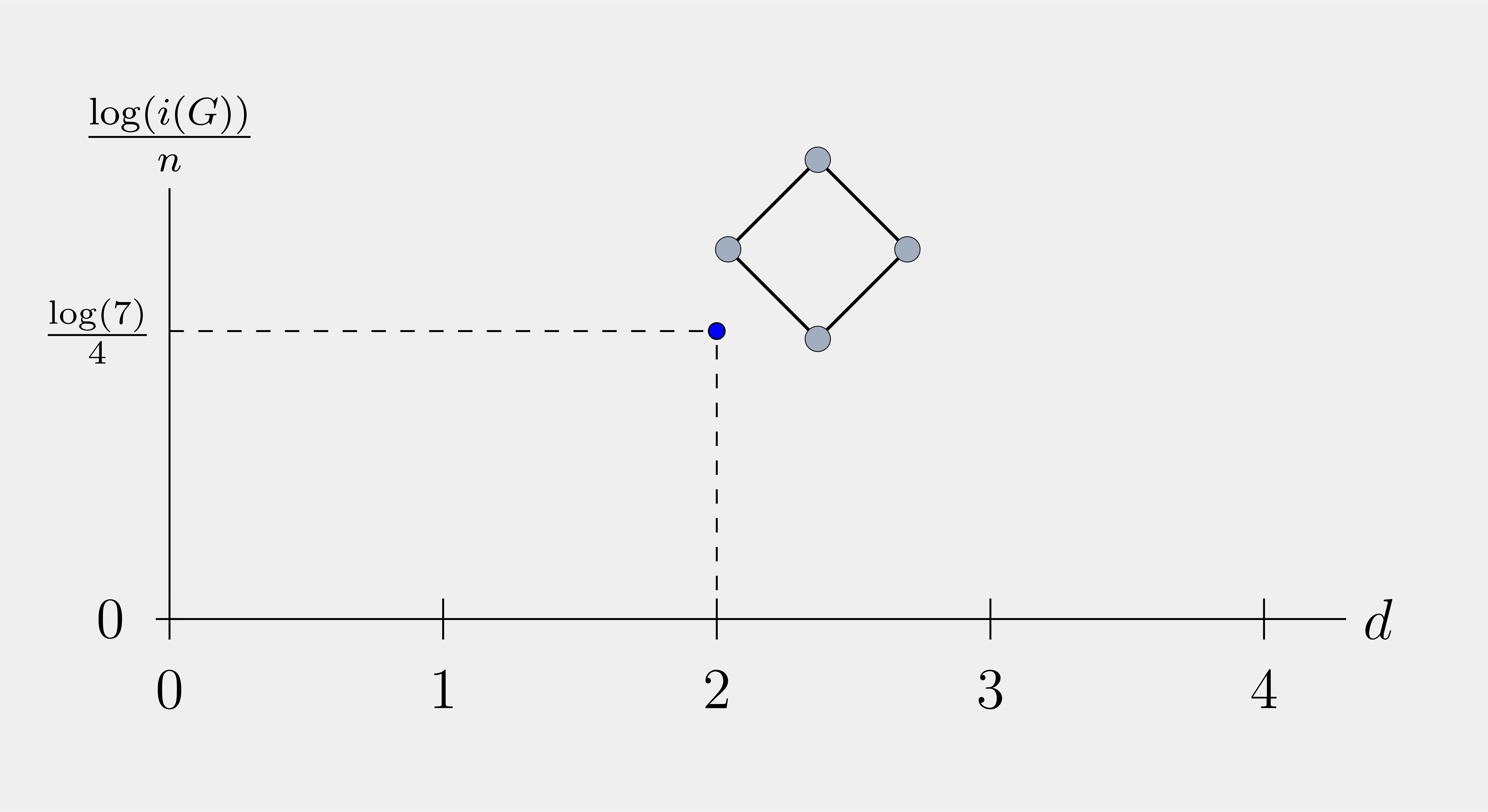
$$f(d) = \frac{d\log(d) - d + 1}{(d-1)^2}$$

What about counting independent sets?

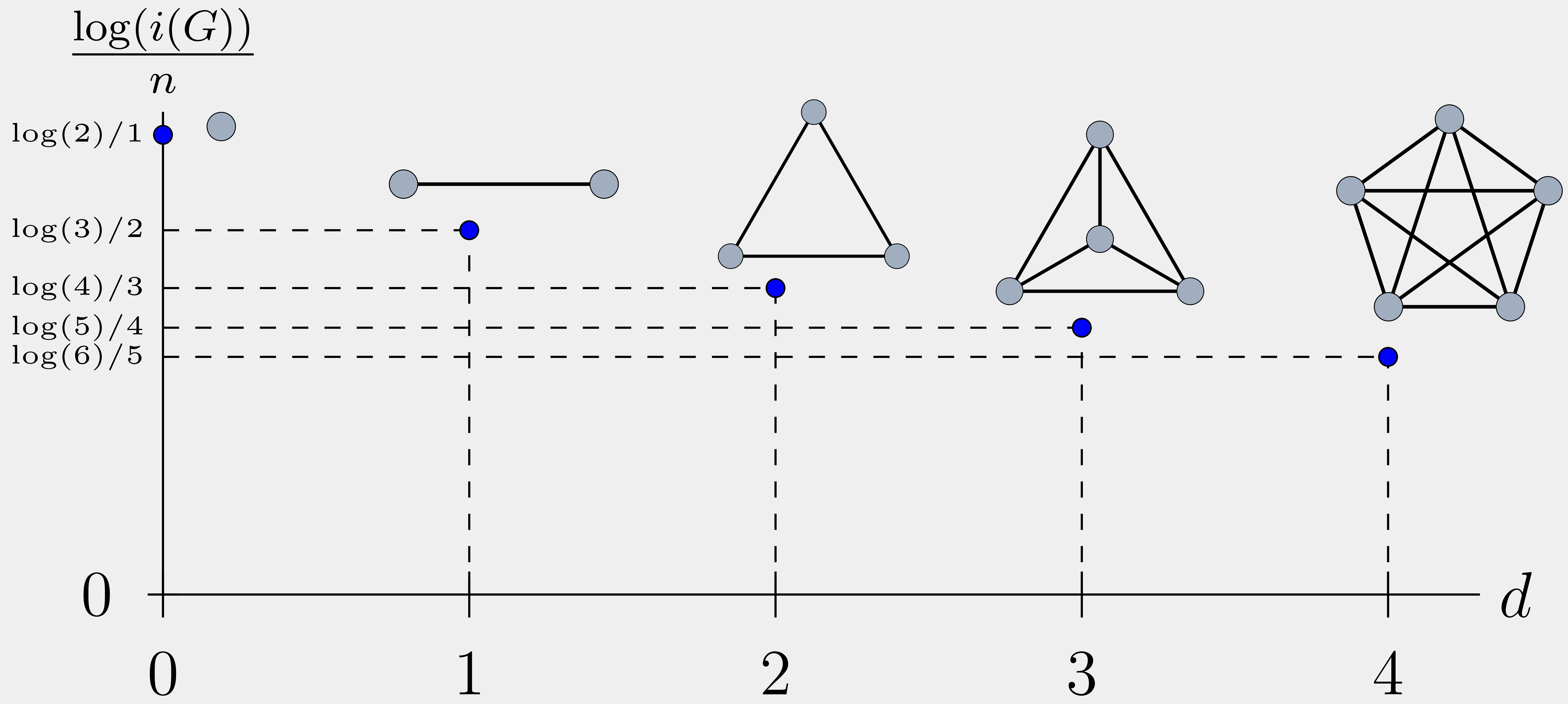
Let $i(G)$ denote the number of distinct independent sets of G.

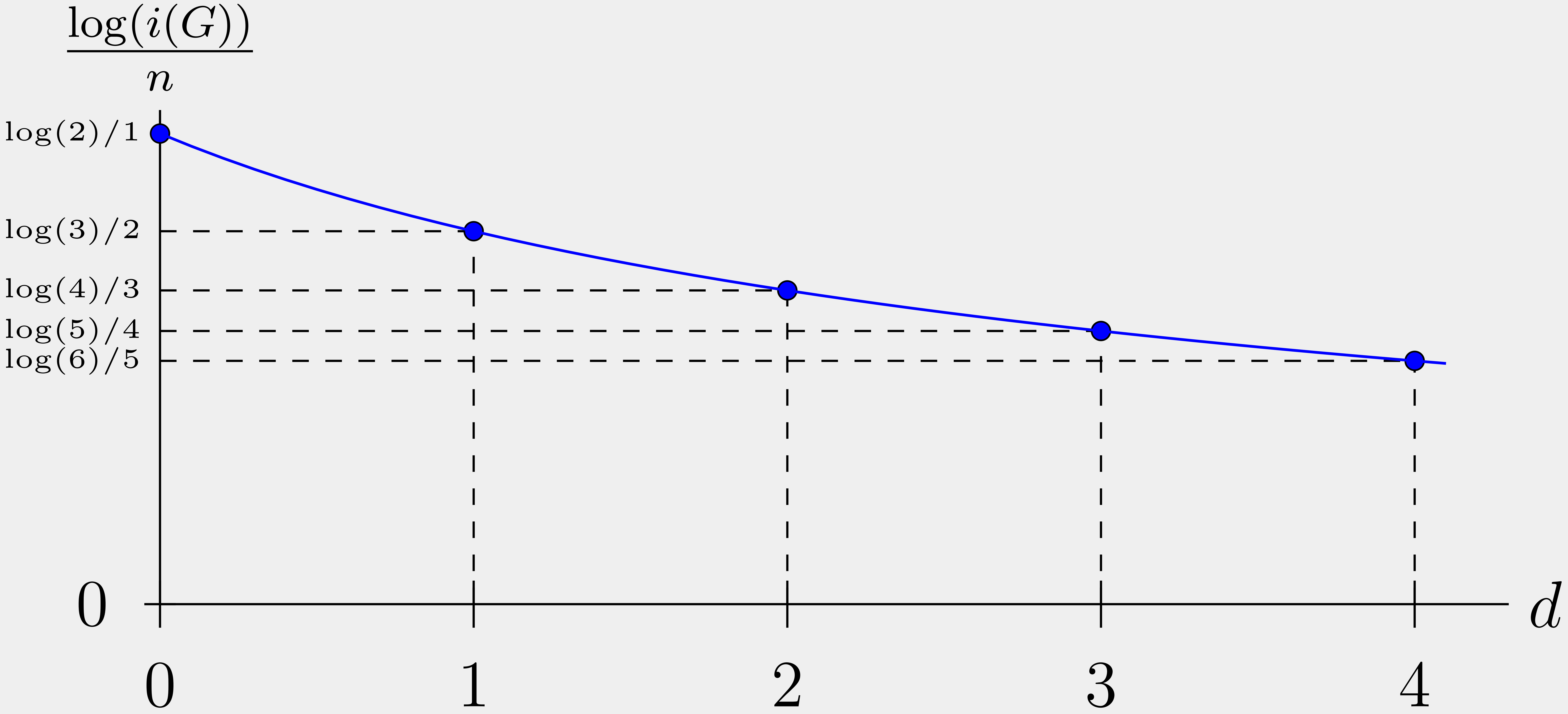


Example: $i(C_4) = 7$



$\frac{\log(i(G))}{n}$
$\frac{\log(7)}{4}$
0
d
0
1
2
3
4


$\frac{\log(i(G))}{n}$
log(2)/1
log(3)/2
log(4)/3
log(5)/4
log(6)/5
0
d
0
1
2
3
4



[Sah, Sawhney, Stoner, Zhao; 2019] $\Longrightarrow \dfrac{\log(i(G))}{|V(G)|} \geq \dfrac{\log(d+2)}{d+1}$.

Define

$$\mathscr{F}(d) = \inf\left\{\log(i(G))/n(G) \,|\, G \text{ triangle-free and average degree at most d}\right\}.$$

Theorem (Cooper, Dutta, Mubayi; 2013):

$$\frac{1}{2400} \cdot \frac{(\log d)^2}{d} \leq \mathscr{F}(d)$$

Theorem (B., van den Heuvel, Kang; 2025):

There is a function f with $f(d) \sim \frac{1}{2} \cdot \frac{(\log d)^2}{d}$ such that $f(d) \leq \mathscr{F}(d)$.

The random graph $G\left(n, d/n\right)$ shows this is asymptotically the right constant.

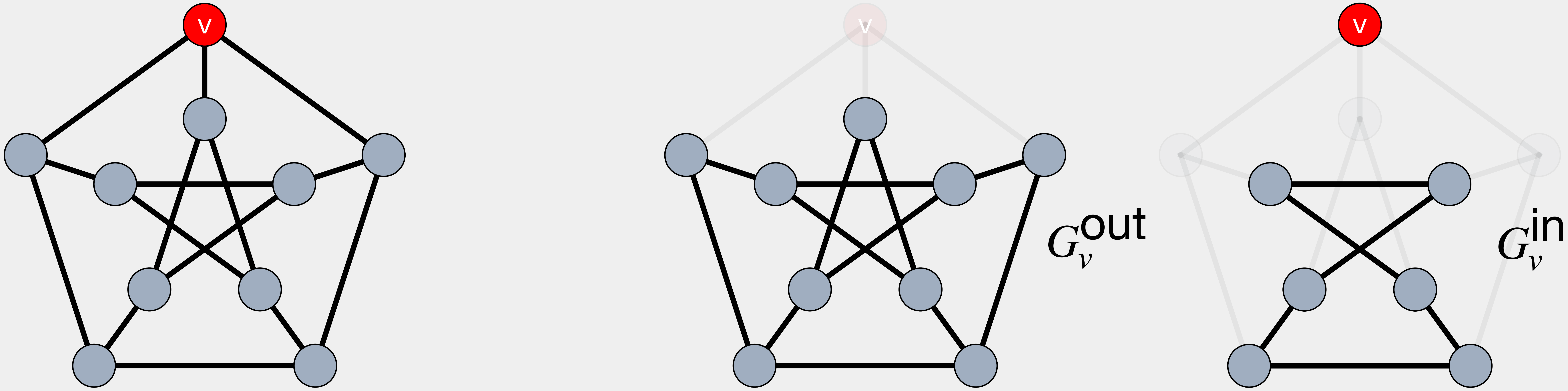
Theorem (B., van den Heuvel, Kang; 2025):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$i(G) \geq \exp(|V(G)| \cdot f(d)).$$

Proof. Take any $v \in V(G)$, then:

$$i(G) = i(G_v^{\text{out}}) + i(G_v^{\text{in}})$$



i) Differentiable. ii) Decreasing. iii) Convex. iv)

Theorem (B., van den Heuvel, Kang; 2025):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$i(G) \geq \exp(|V(G)| \cdot f(d)).$$

Proof. Take any $v \in V(G)$, then:

$$i(G) = i(G_v^{\text{out}}) + i(G_v^{\text{in}})$$
$$\geq \exp(|V(G_v^{\text{out}})| \cdot f(d(G_v^{\text{out}}))) + \exp(|V(G_v^{\text{in}})| \cdot f(d(G_v^{\text{in}})))$$

Pick v uniformly at random then the RHS in expectation is at least:

$$e^{|V(G)| \cdot f(d)} \cdot \left[e^{-df'(d) - f(d)} + e^{(d-d^2)f'(d) - (d+1)f(d)} \right]$$

i) Differentiable. ii) Decreasing. iii) Convex. iv)

Theorem (B., van den Heuvel, Kang; 2025):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$i(G) \geq \exp(|V(G)| \cdot f(d)).$$

Proof. Take any $v \in V(G)$, then:

$$\begin{aligned} i(G) &= i(G_v^{\text{out}}) + i(G_v^{\text{in}}) \\ &\geq \exp(|V(G_v^{\text{out}})| \cdot f(d(G_v^{\text{out}}))) + \exp(|V(G_v^{\text{in}})| \cdot f(d(G_v^{\text{in}}))) \end{aligned}$$

Pick v uniformly at random then the RHS in expectation is at least:

$$e^{|V(G)| \cdot f(d)} \cdot \left[e^{-df'(d)-f(d)} + e^{(d-d^2)f'(d)-(d+1)f(d)} \right]$$

Which is at least $e^{|V(G) \cdot f(d)|}$ by iv).

i) Differentiable. ii) Decreasing. iii) Convex. iv) $e^{-df'(d)-f(d)} + e^{(d-d^2)f'(d)-(d+1)f(d)} \geq 1$

Theorem (B., van den Heuvel, Kang; 2025):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$i(G) \geq \exp(|V(G)| \cdot f(d)).$$

i) Differentiable. ii) Decreasing. iii) Convex. iv) $e^{-df'(d)-f(d)} + e^{(d-d^2)f'(d)-(d+1)f(d)} \geq 1$

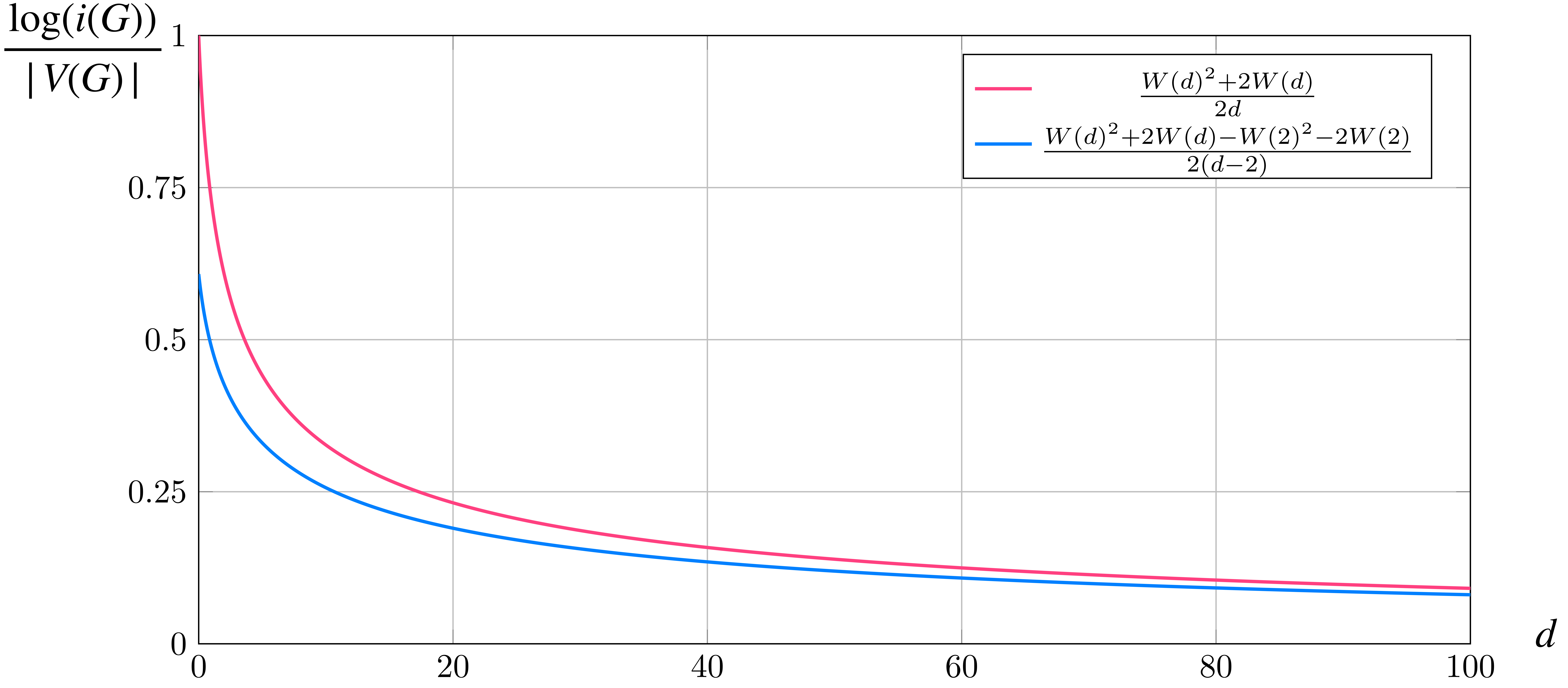
Theorem (B., van den Heuvel, Kang; 2025):

Let $y \mapsto W(y)$ denote the inverse of $x \mapsto xe^x$, then f satisfies i), ii), iii), iv):

$$f(d) = \frac{W(d)^2 + 2W(d) - W(2)^2 - 2W(2)}{2(d-2)}$$

The random graph $G\left(n, d/n\right)$ shows that there exist G with

$$\log(i(G))/|V(G)| \leq \frac{W(d)^2 + 2W(d)}{2d} \qquad +\varepsilon$$

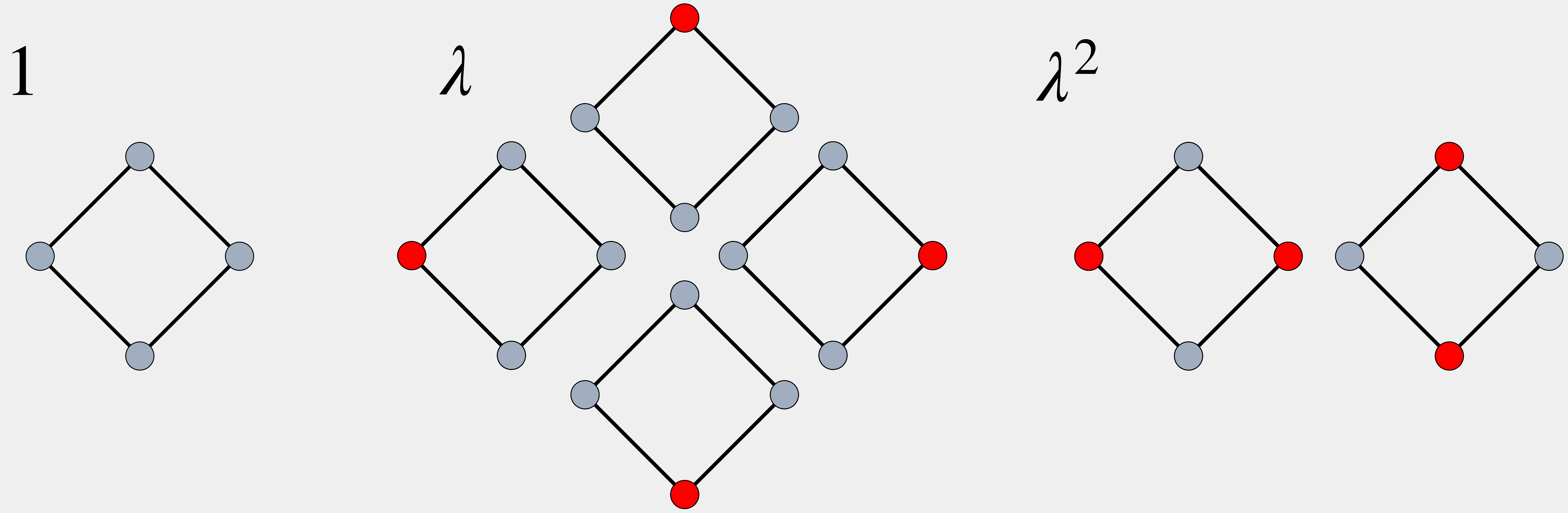


$$\left[(\log d)^2 - 2\log\log d \cdot \log d + 2\log d + (\log\log d)^2 + o(1)\right]/(2d)$$

$$\left[(\log d)^2 - 2\log\log d \cdot \log d + 2\log d + (\log\log d)^2 - 1.28\right]/(2d)$$

The independent set polynomial.

Let $\lambda \geq 0$ and let $Z_G(\lambda)$ denote the weighted count of independent sets:



Example: $Z_{C_4}(\lambda) = 1 + 4\lambda + 2\lambda^2$

Theorem (B., van den Heuvel, Kang; 2025):

Suppose f satisfies i), ii), iii), iv), then for any triangle-free G with average degree d:

$$Z_G(\lambda) \geq \exp(|V(G)| \cdot f(d)).$$

i) Differentiable. ii) Decreasing. iii) Convex.

iv) $$e^{-df'(d)-f(d)} + \lambda \cdot e^{(d-d^2)f'(d)-(d+1)f(d)} \geq 1$$

Suppose f_λ is a family of functions such that for each fixed λ, f_λ satisfies i), ii), iii), iv)

Then for fixed d and any triangle-free G with average degree at most d:

$$\alpha(G)/|V(G)| \geq \limsup_{\lambda \to \infty} \left[f_\lambda(d)/\log(\lambda) \right]$$

Question: Is it possible to improve on Shearer in this way?