

# A limit theorem for Shannon capacity from lattice packings

Pjotr Buys<sup>1</sup>, Sven Polak<sup>2</sup>, and Jeroen Zuiddam<sup>1</sup>

1. University of Amsterdam
2. Tilburg University

## Question

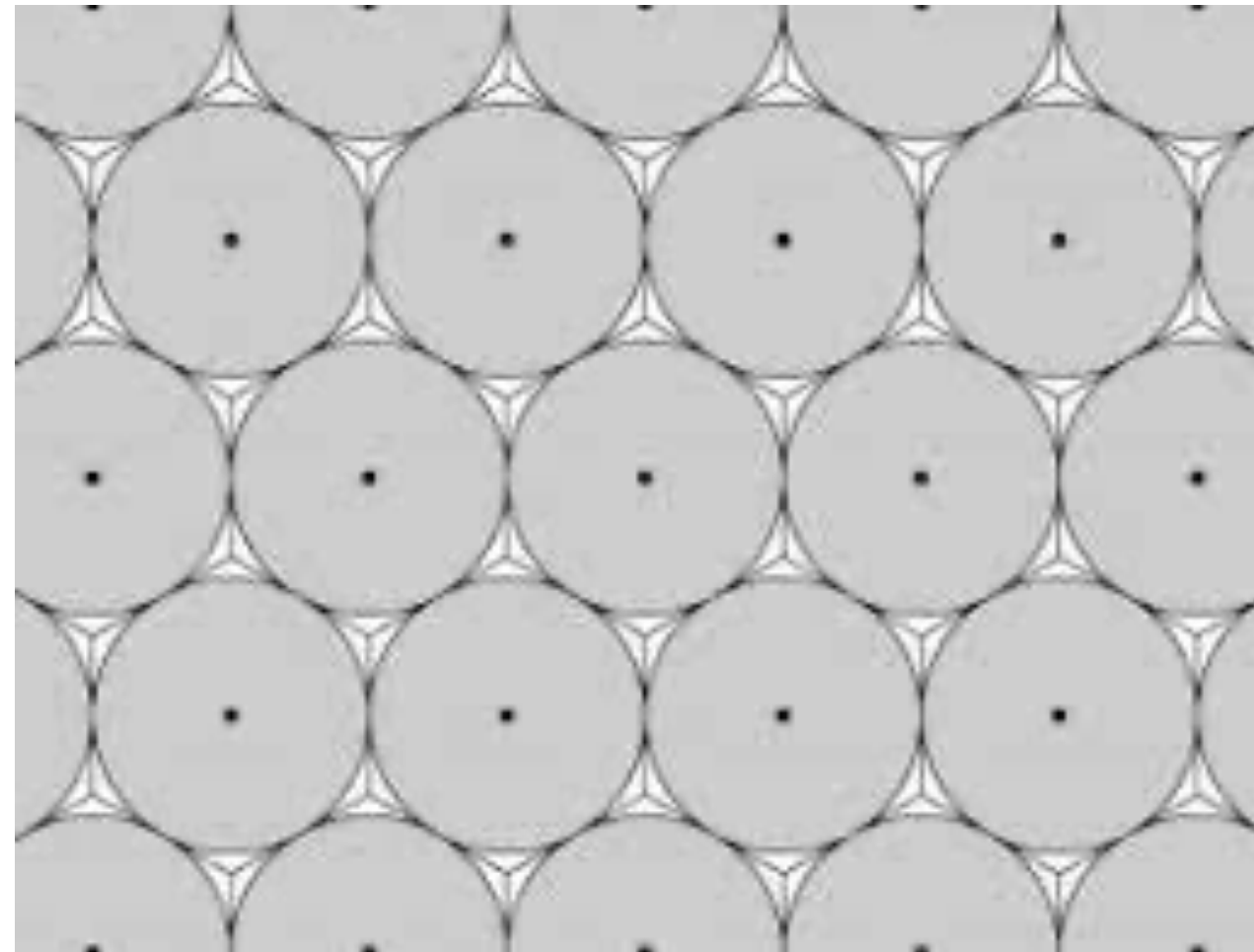
What is the maximal **density** of  $S \subseteq \mathbb{R}^n$  satisfying  $|x - y| \geq 1$  for distinct  $x, y \in S$ .

$n = 1$



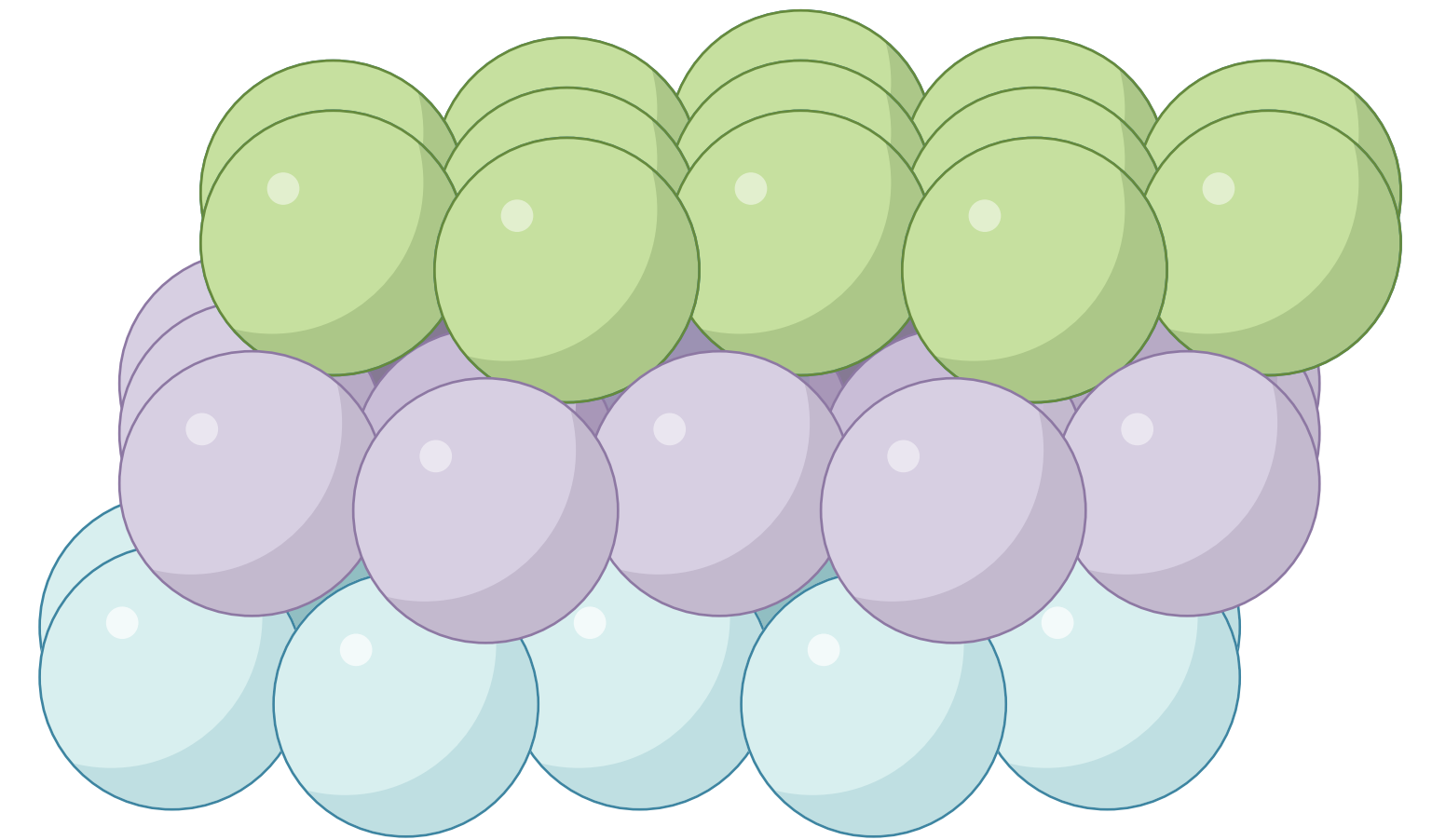
Trivial

$n = 2$



[Thue; 1911]

$n = 3$



[Hales; 1998]

Known for  $n = 8$  [Viazovska; 2016] and  $n = 24$  [Viazovska et al.; 2016].

Asymptotically the maximal density  $\theta(n)$  satisfies:

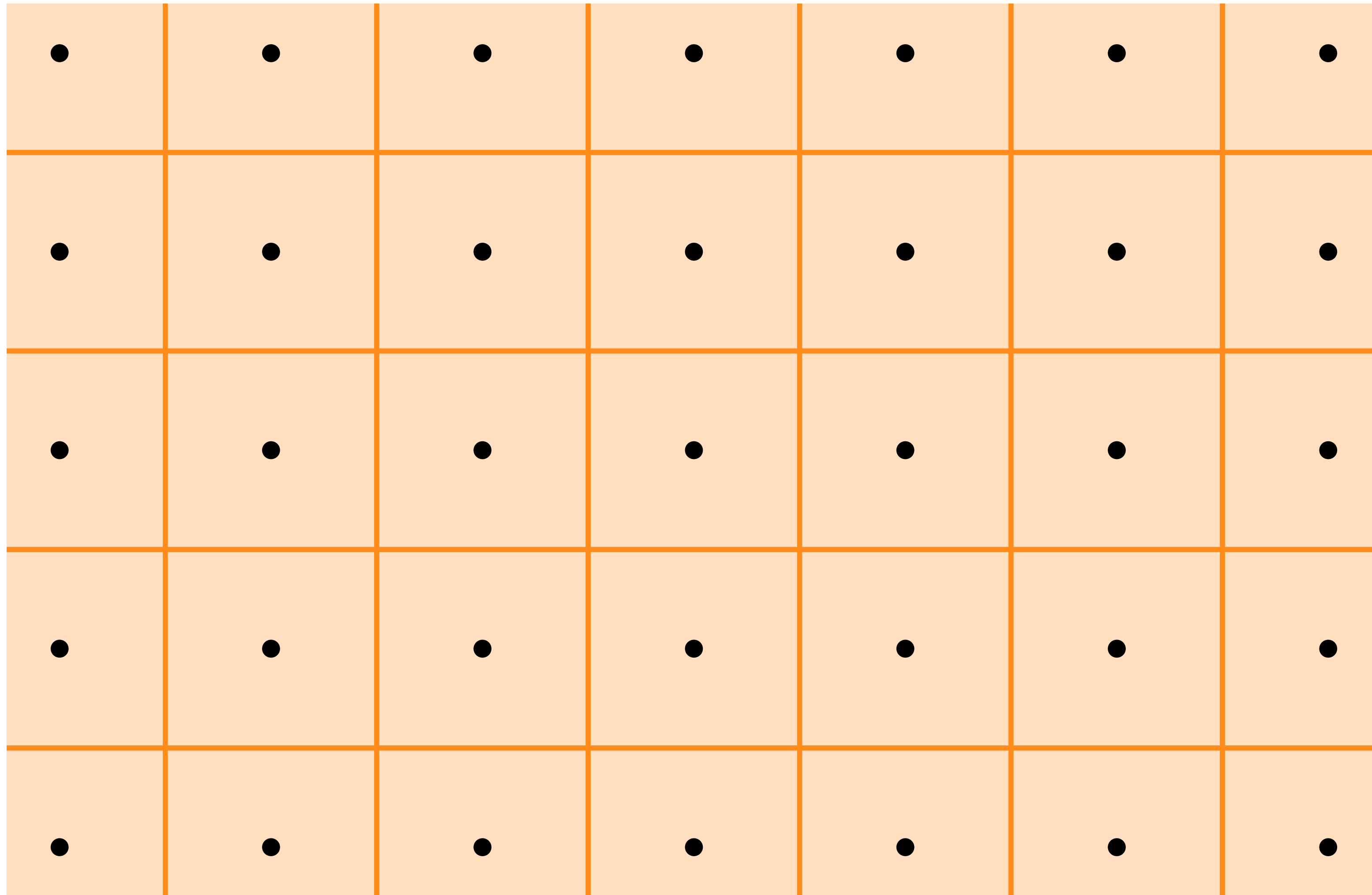
$$c \cdot n^2 \cdot 2^{-n} \leq \theta(n) \leq 2^{-(0.599+o(1)) \cdot n}$$

[Klartag; 2025]

[Kabatjanskiĭ, Levenšteĭn; 1978]

## Question

What is the maximal **density** of  $S \subseteq \mathbb{R}^n$  satisfying  $|x - y|_\infty \geq 1$  for distinct  $x, y \in S$ .

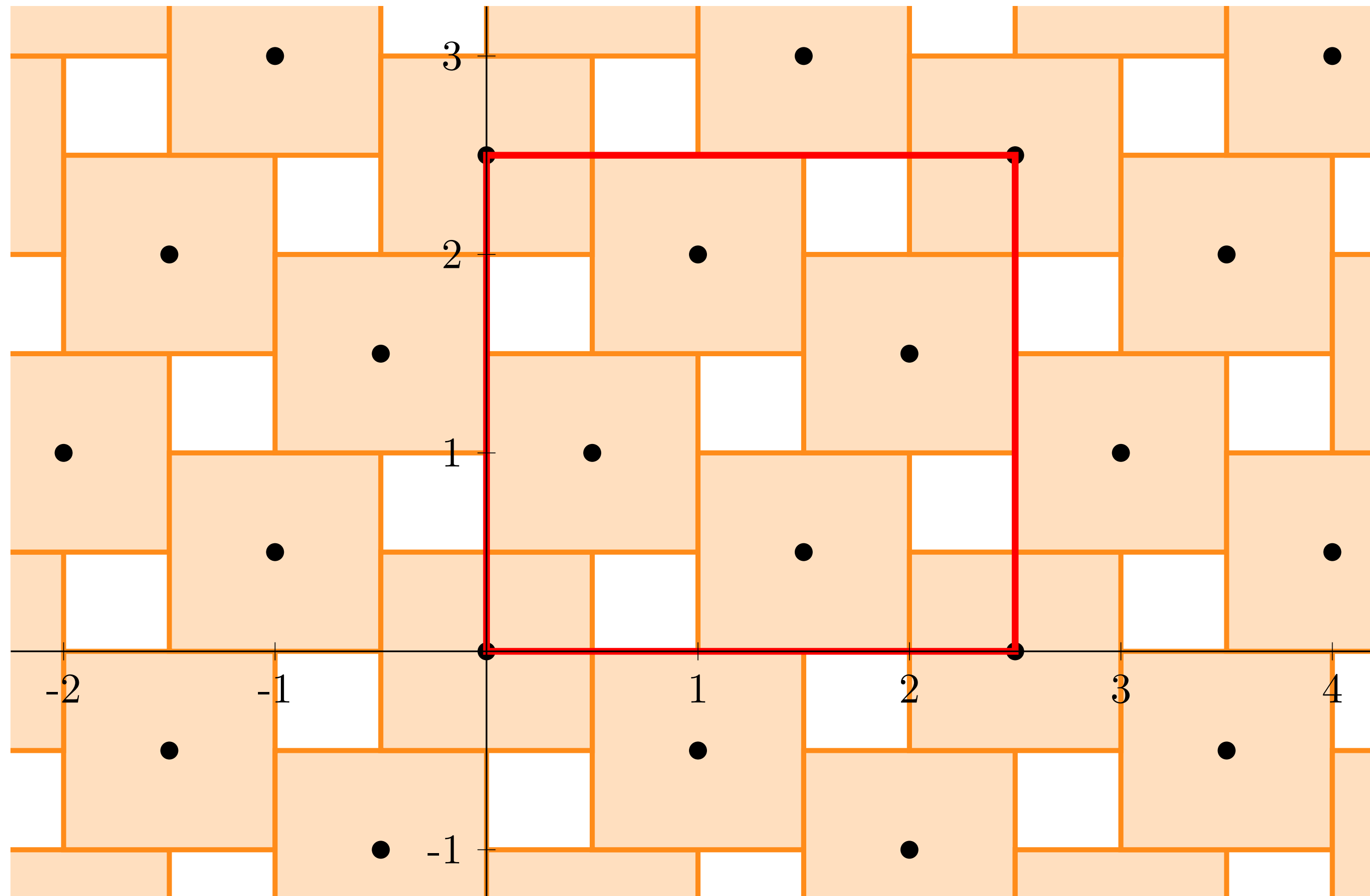


$$|x - y|_\infty = \max |x_i - y_i|$$

## Question

Let  $r \geq 1$ . What is the maximal size of  $S \subseteq \mathbb{R}^n$  satisfying

- $|x - y|_\infty \geq 1$  for distinct  $x, y \in S$ ;
- $S$  is  $r$ -periodic in every unit direction.

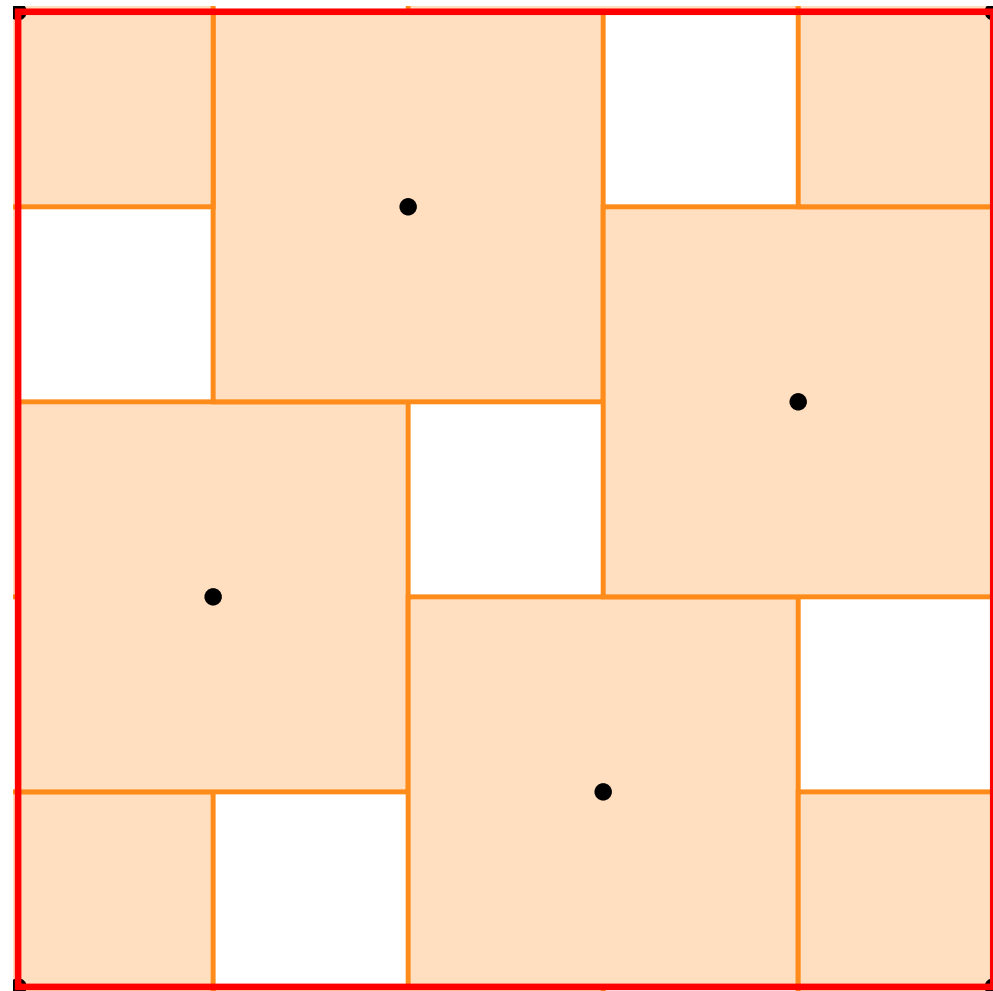


$$r = 5/2$$

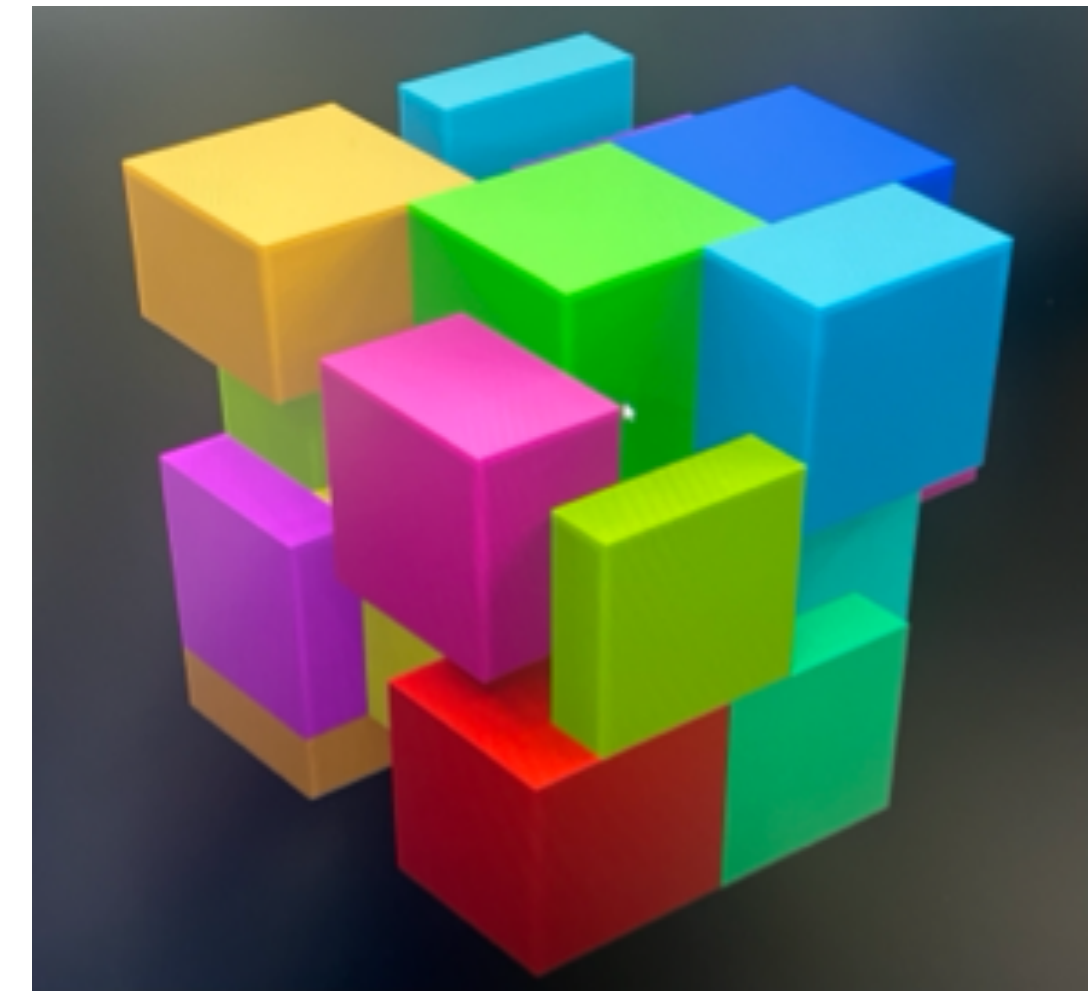
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$$r = 8/3$$

## Definition

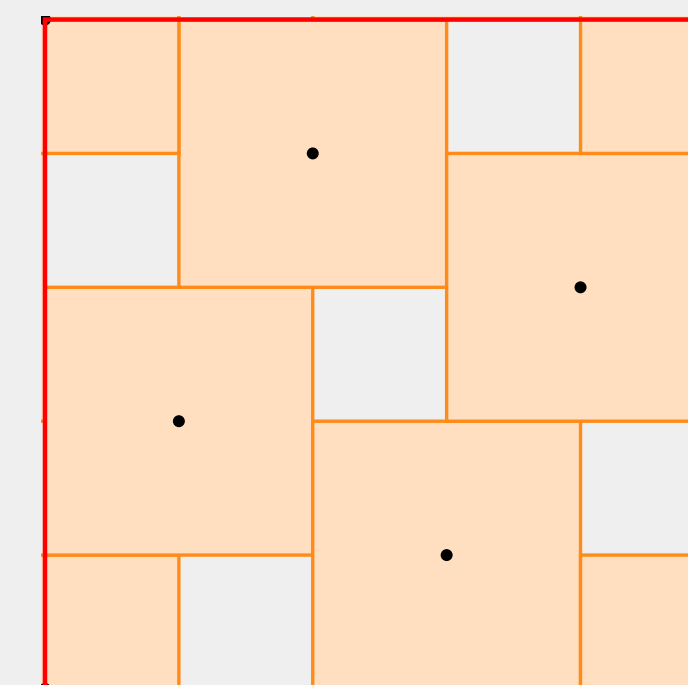
For  $r \in \mathbb{R}_{\geq 1}$  and  $n \in \mathbb{N}$  let  $\alpha_n(r)$  denote the maximum number of  $n$ -dimensional hypercubes of side length one that fit in an  $n$ -dimensional torus of side length  $r$ .

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## Observations

- The function  $r \mapsto \alpha_n(r)$  is nondecreasing.
- For  $r \in [1, 2)$  we have  $\alpha_n(r) = 1$ .
- For integer  $r \geq 1$  we have  $\alpha_n(r) = r^n$ .
- The volume of the torus is  $r^n$  and thus  $\alpha_n(r) \leq r^n$ .



## Definition

For  $r \in \mathbb{R}_{\geq 1}$  we define:

$$\Theta(r) = \sup_{n \geq 1} \alpha_n(r)^{1/n}$$

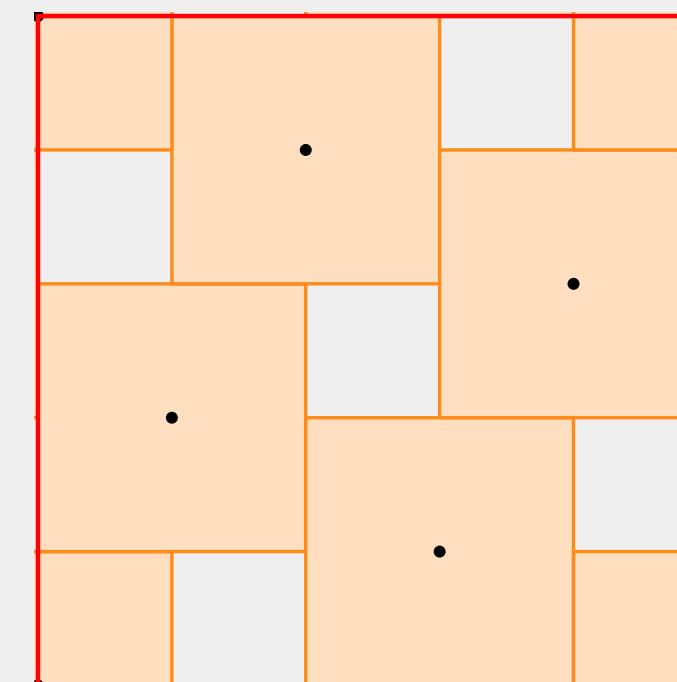
Note that  $\Theta$  is nondecreasing with  $\Theta(r) \leq r$  with equality for integer  $r$ .

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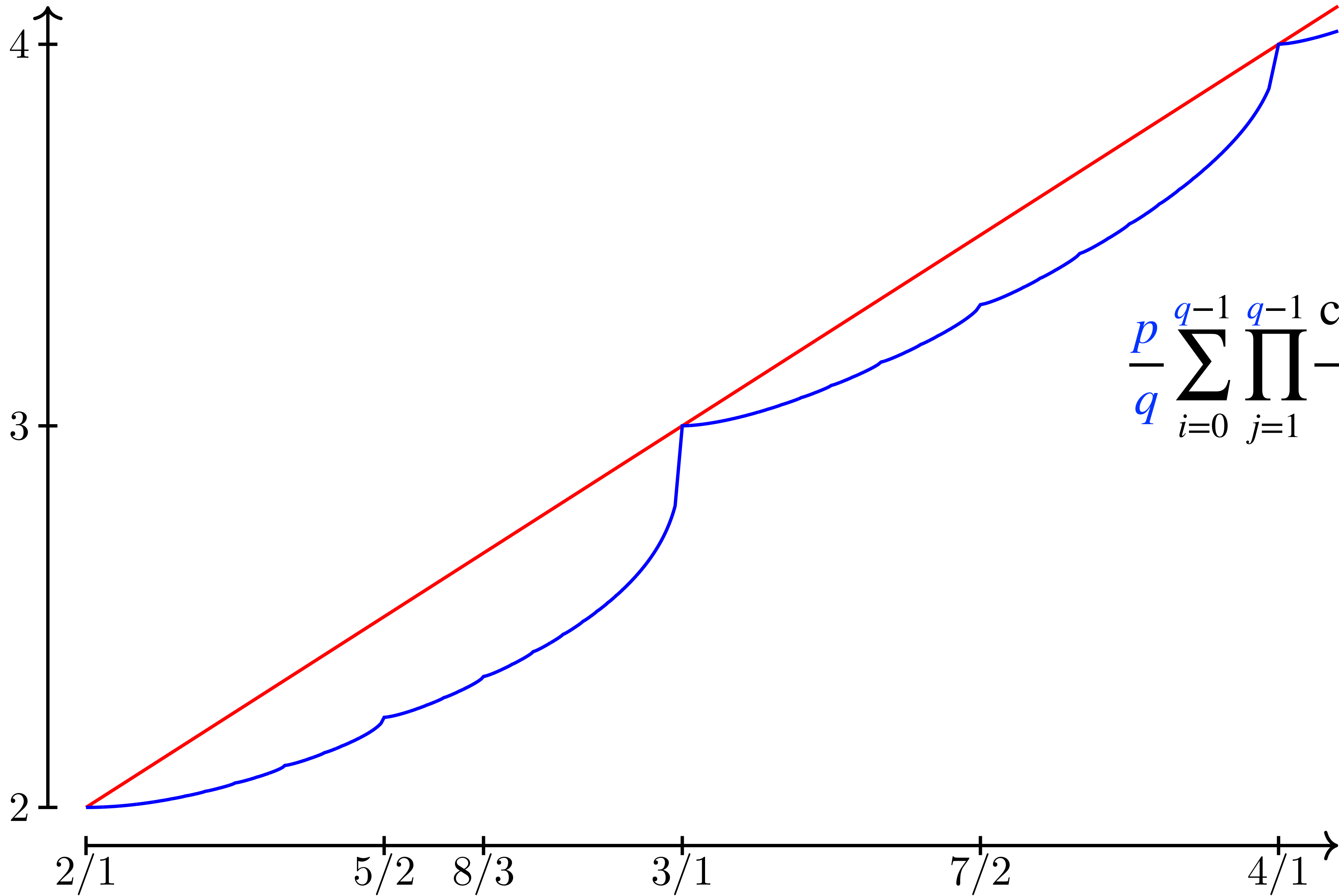
**Theorem** [Lovász; 1979]

$$\Theta(5/2) = \sqrt{5}$$



**Theorem** [Bachoc, Pêcher and Thiéry; 2013]

$$\Theta\left(\frac{p}{q}\right) \leq \frac{p}{q} \sum_{i=0}^{q-1} \prod_{j=1}^{q-1} \frac{\cos\left(\frac{2i\pi}{q}\right) - \cos\left(\lfloor \frac{pj}{q} \rfloor \frac{2\pi}{p}\right)}{1 - \cos\left(\lfloor \frac{pj}{q} \rfloor \frac{2\pi}{p}\right)}$$



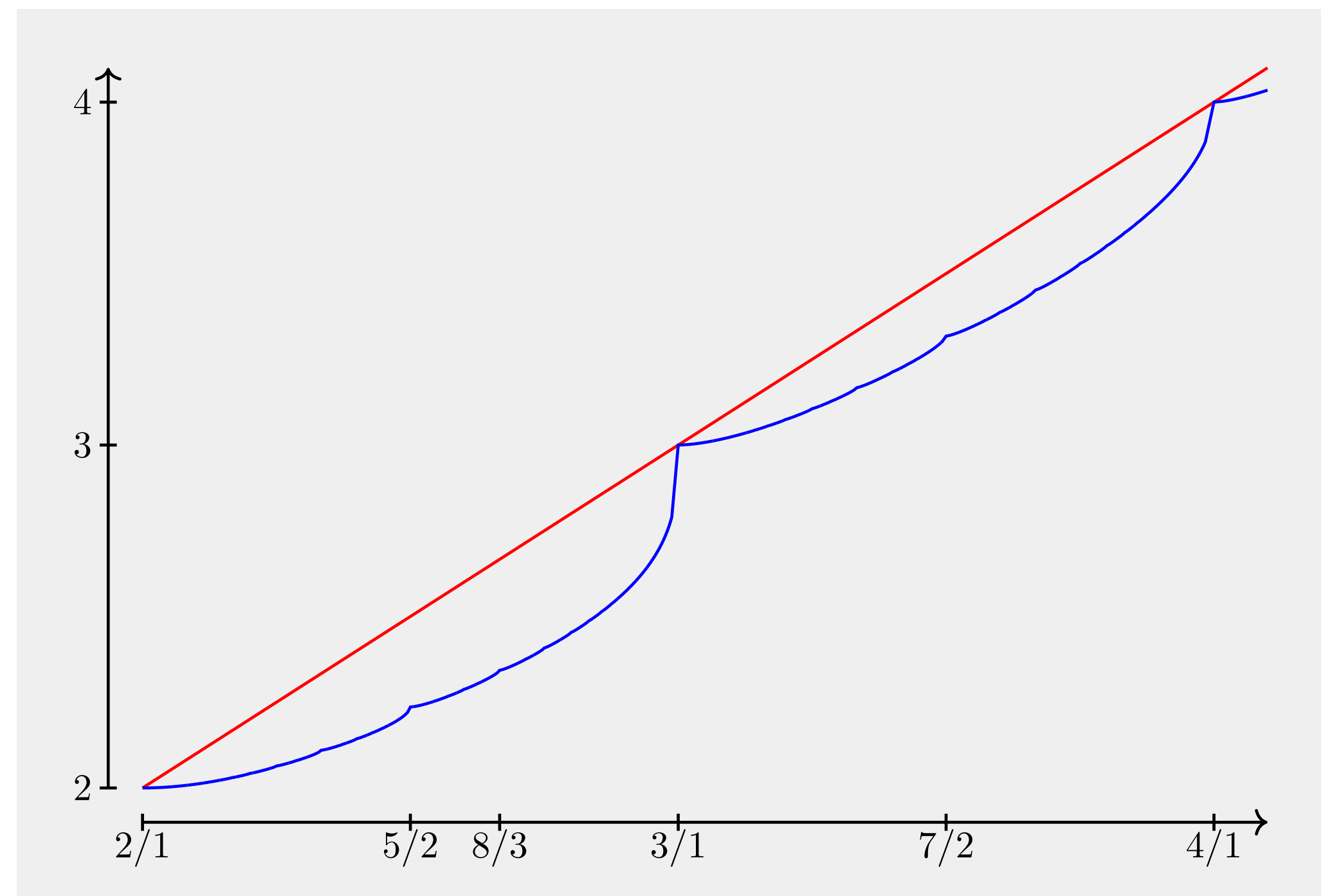
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## Theorem [Bohman; 2003]

We have  $\lim_{m \rightarrow \infty} (m + 1/2) - \Theta(m + 1/2) = 0$  (with  $m \in \mathbb{N}$ .)

## Theorem [B., Polak, Zuiddam; 2025]

We have  $\lim_{r \rightarrow \infty} r - \Theta(r) = 0$ .

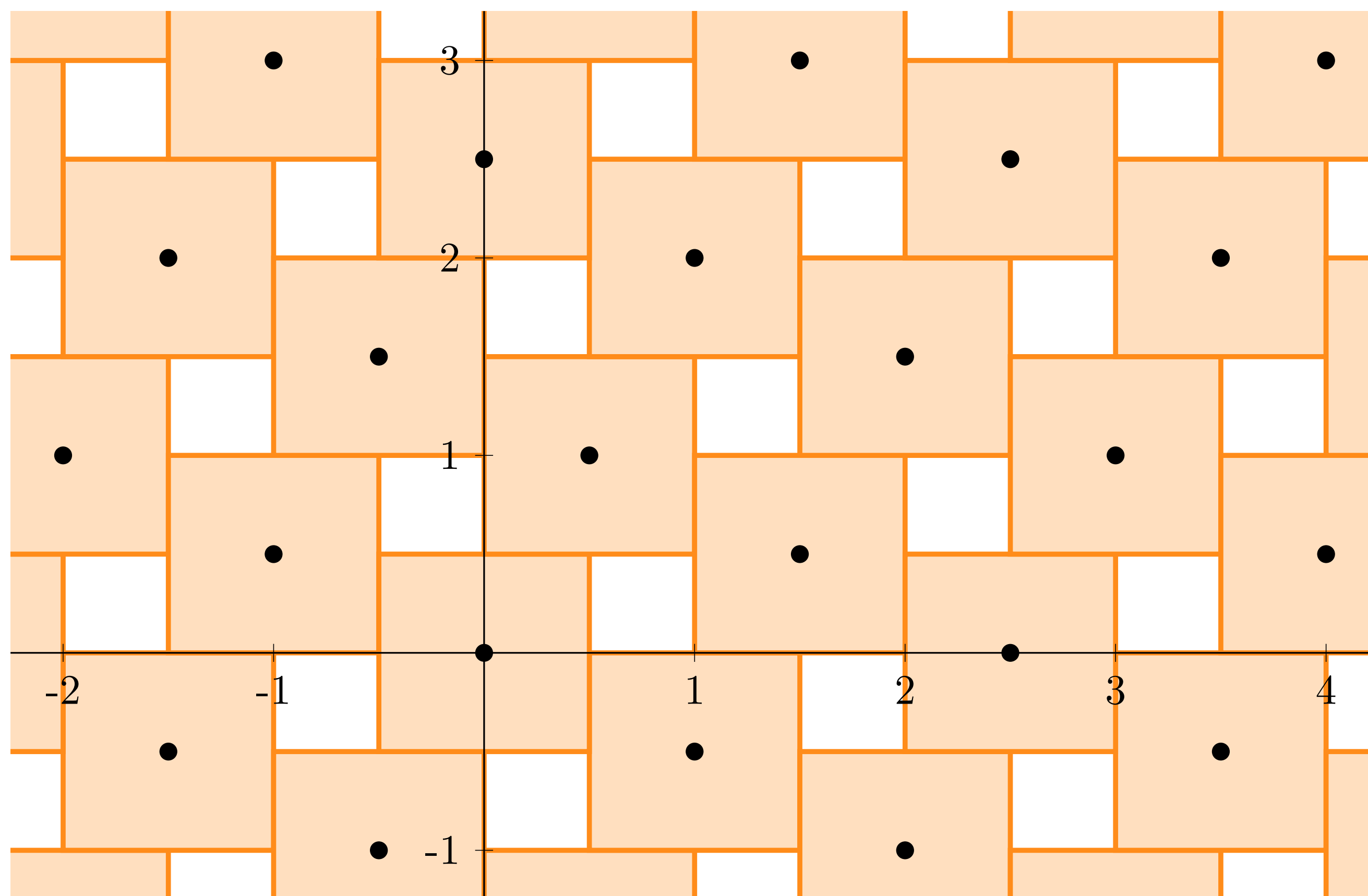


## Definition

For an (invertible) matrix  $A \in \mathbb{R}^{n \times n}$ , let  $\mathcal{L}(A)$  denote the lattice spanned by its columns.

$$\mathcal{L}(A) = \{Av : v \in \mathbb{Z}^n\}$$

For a lattice  $\Lambda$  we let  $\lambda_\infty(\Lambda) = \min\{|v|_\infty : v \in \Lambda, v \neq 0\}$ .

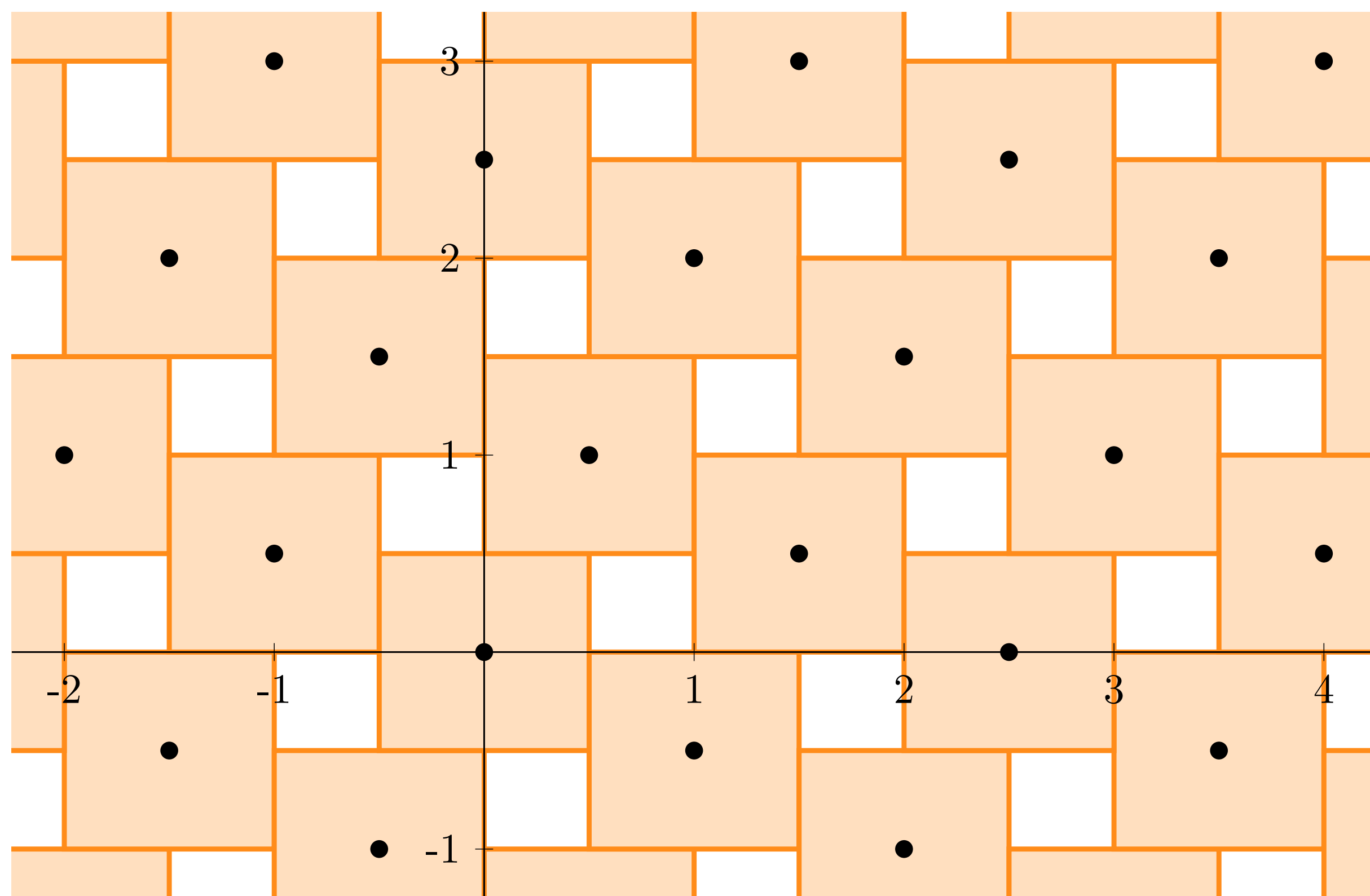


$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$\lambda_\infty(\mathcal{L}(A)) = 1$$

# Lemma

Let  $r \in \mathbb{R}_{\geq 1}$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{Z}^{n \times n}$  such that



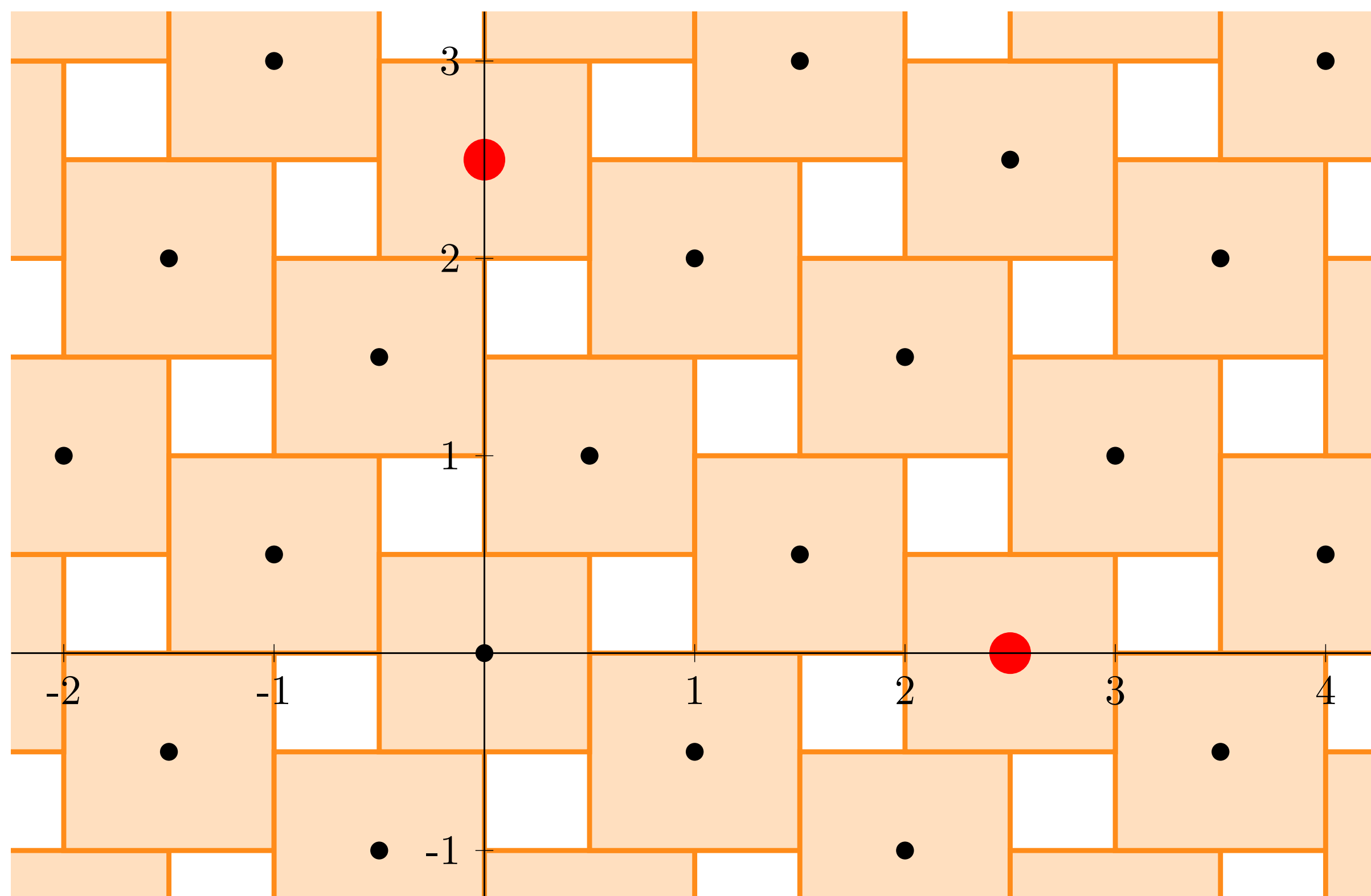
$\mathcal{L}(A)$

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad r = 5/2$$

# Lemma

Let  $r \in \mathbb{R}_{\geq 1}$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{Z}^{n \times n}$  such that

- $AB = r I_n$



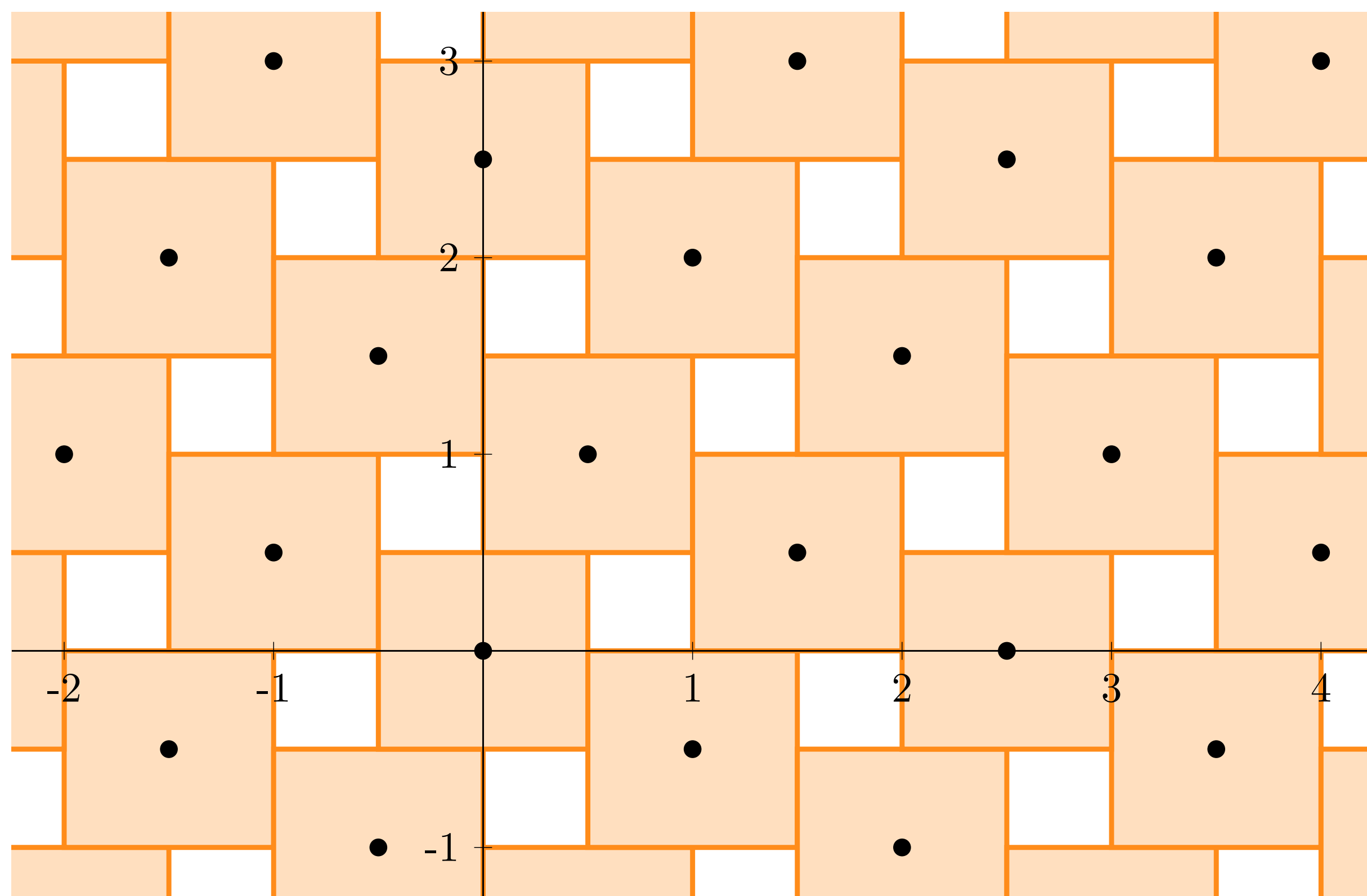
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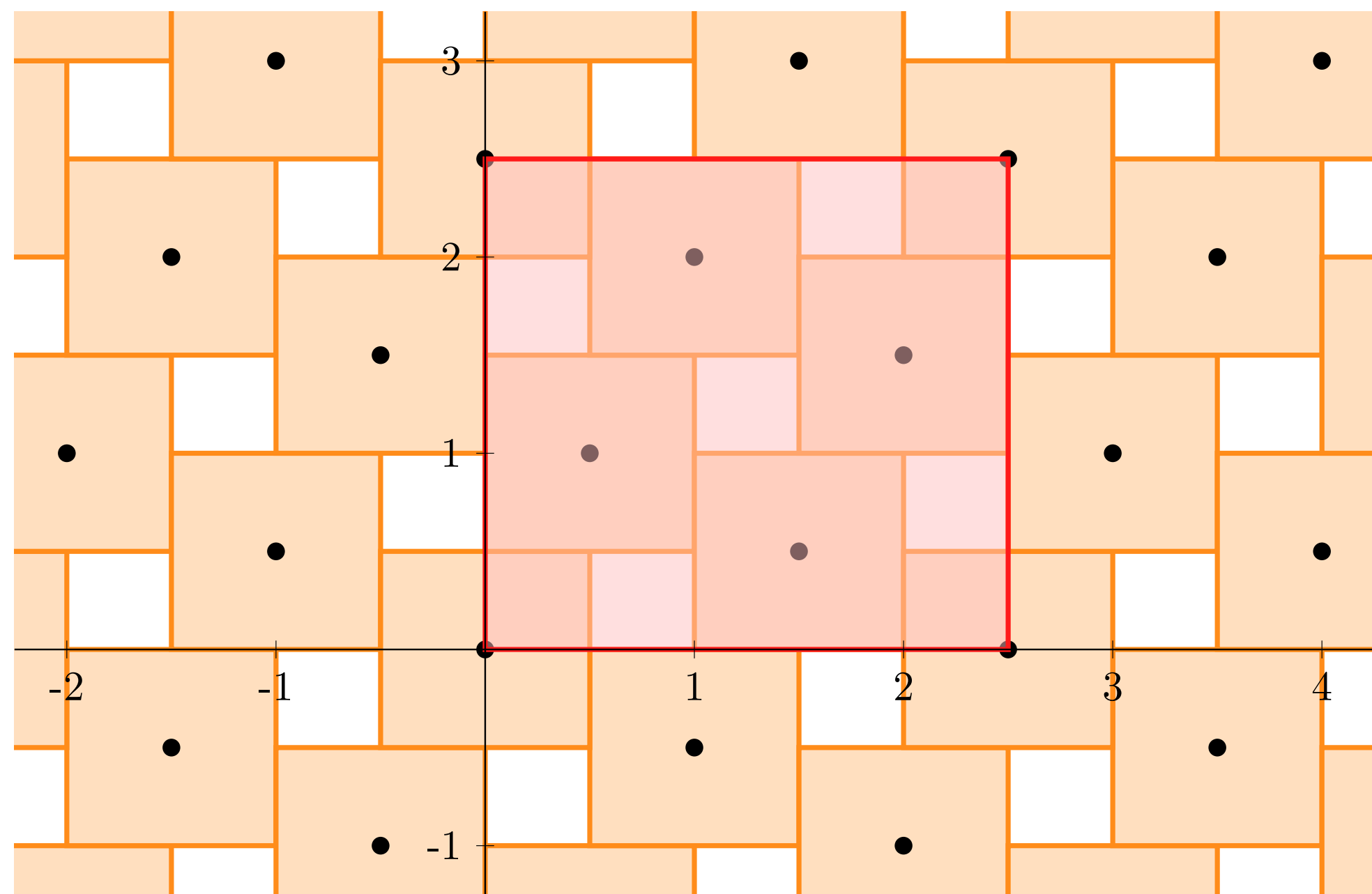
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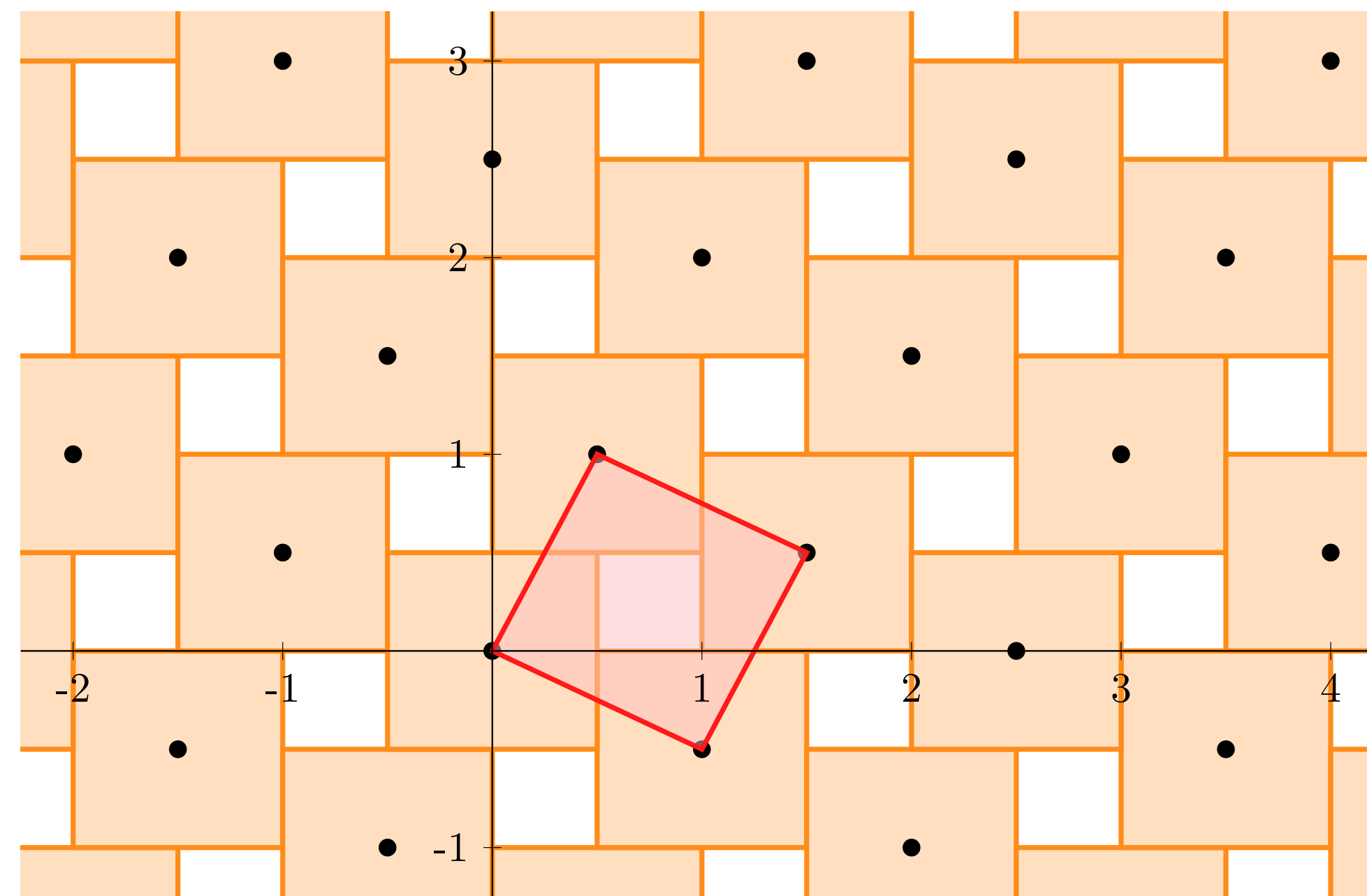
- $AB = r I_n$
- $\lambda_\infty(\mathcal{L}(A)) \geq 1$

Then  $\alpha_n(r) \geq |\det(B)|$ .

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad r = 5/2$$



Volume torus is  $r^n = \det(rI_n)$



Volume fundamental domain is  $|\det(A)|$

## Lemma

Let  $r \in \mathbb{R}_{\geq 1}$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{Z}^{n \times n}$  such that

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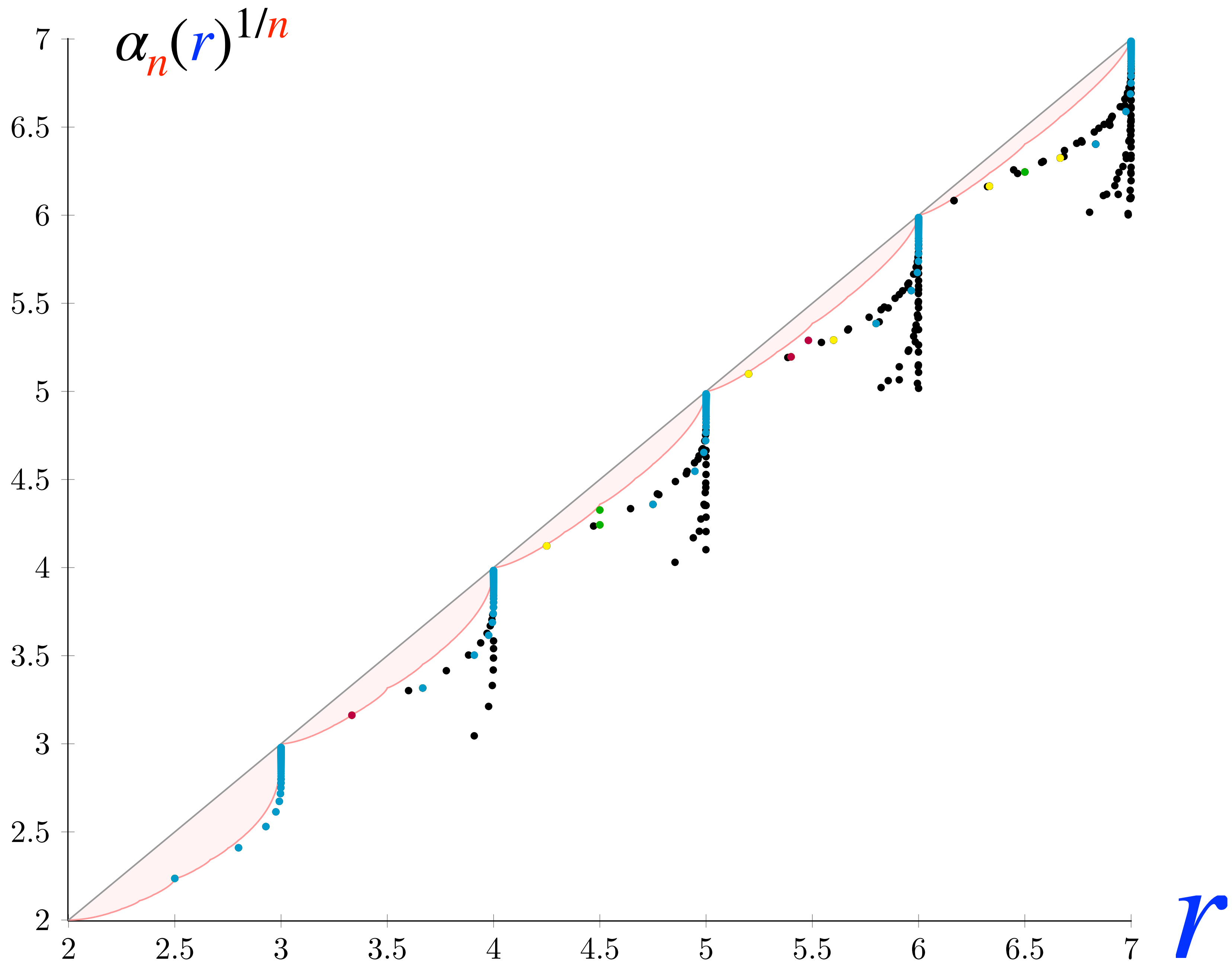
Then  $\alpha_n(r) \geq |\det(B)|$ .

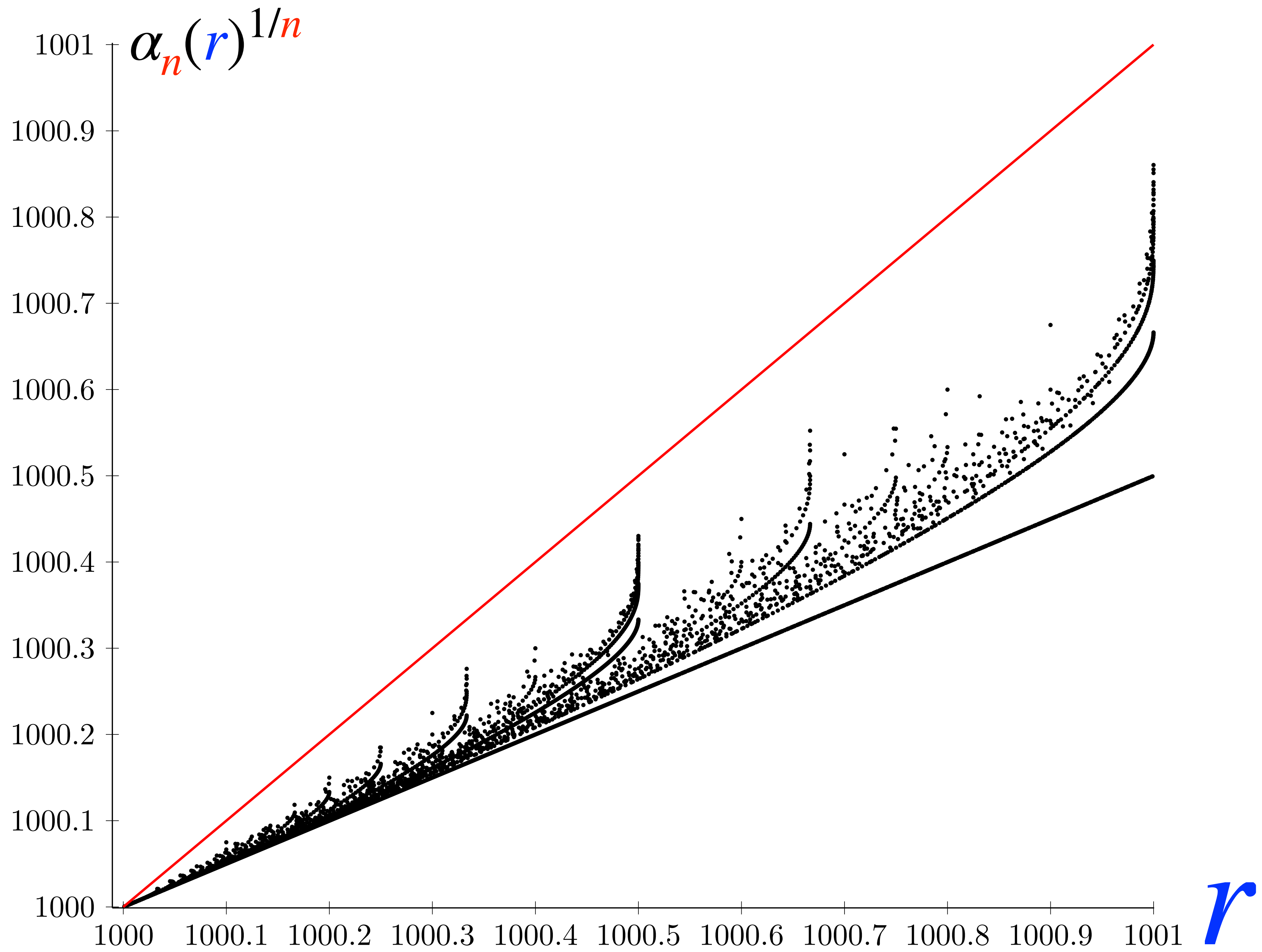
## Example

For  $n = 5$  and parameters  $k, b, r, s$  let

$$B = \begin{pmatrix} b^4 k + s & -r & -br & -b^2 r & -b^3 r \\ b^3 k & b^4 k + s & -r & -br & -b^2 r \\ b^2 k & b^3 k & b^4 k + s & -r & -br \\ bk & b^2 k & b^3 k & b^4 k + s & -r \\ k & bk & b^2 k & b^3 k & b^4 k + s \end{pmatrix} \quad r = \frac{k(b^5 k + bs + r)^5 + rs^5}{bk(b^5 k + bs + r)^4 + rs^4}$$

and  $A = rB^{-1}$ . These satisfy the assumptions of the lemma (under some constraints).





## Lemma

Let  $r \in \mathbb{R}_{\geq 1}$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{Z}^{n \times n}$  such that

- $AB = r I_n$
- $\lambda_\infty(\mathcal{L}(A)) \geq 1$

Then  $\alpha_n(r) \geq |\det(B)|$ .

## Lemma

If all principal minors of  $A - I_n$  are nonnegative, then  $\lambda_\infty(\mathcal{L}(A)) \geq 1$ .

## Example

$$A = \begin{pmatrix} 1 & \frac{8}{979} & \frac{190}{979} \\ -\frac{285}{979} & 1 & \frac{8}{979} \\ -\frac{12}{979} & -\frac{285}{979} & 1 \end{pmatrix}$$

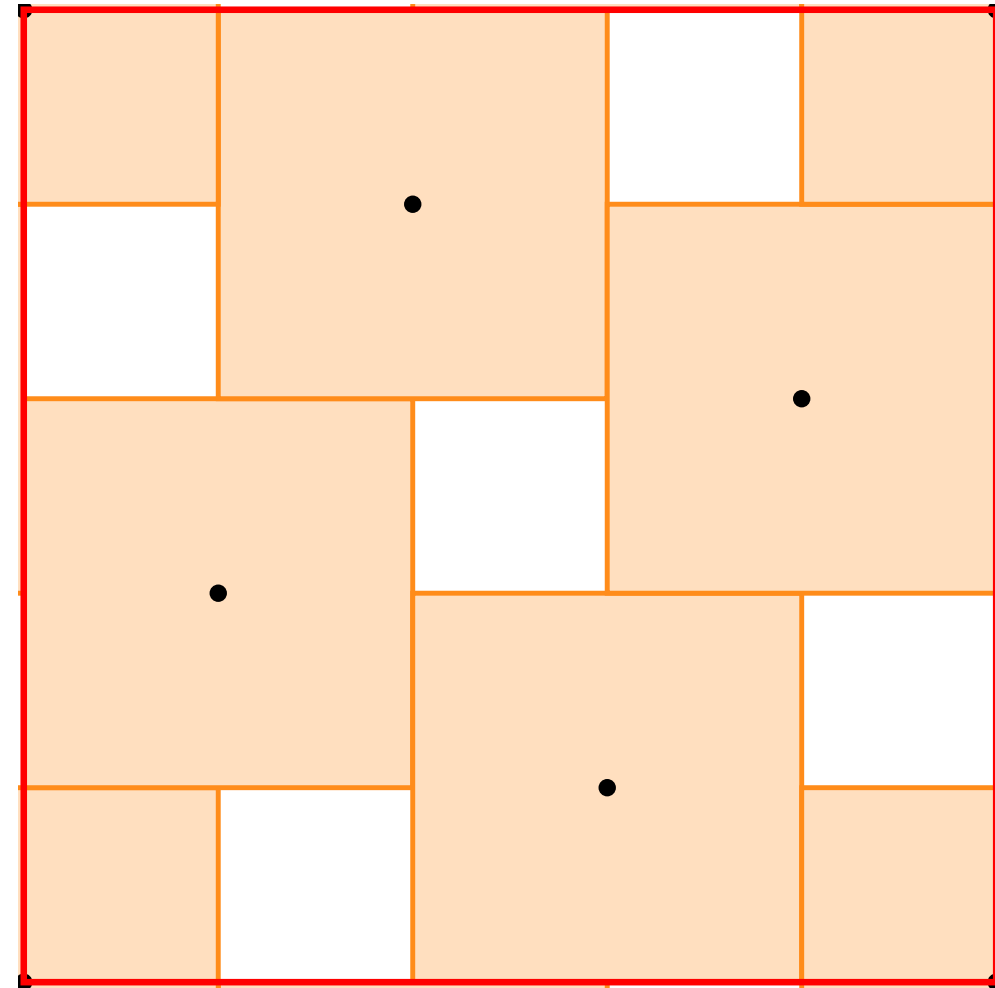
$$\det(A - I_3) = \frac{15431982}{938313739}$$

$$\det \begin{pmatrix} 0 & \frac{8}{979} \\ -\frac{285}{979} & 0 \end{pmatrix} \geq 0$$

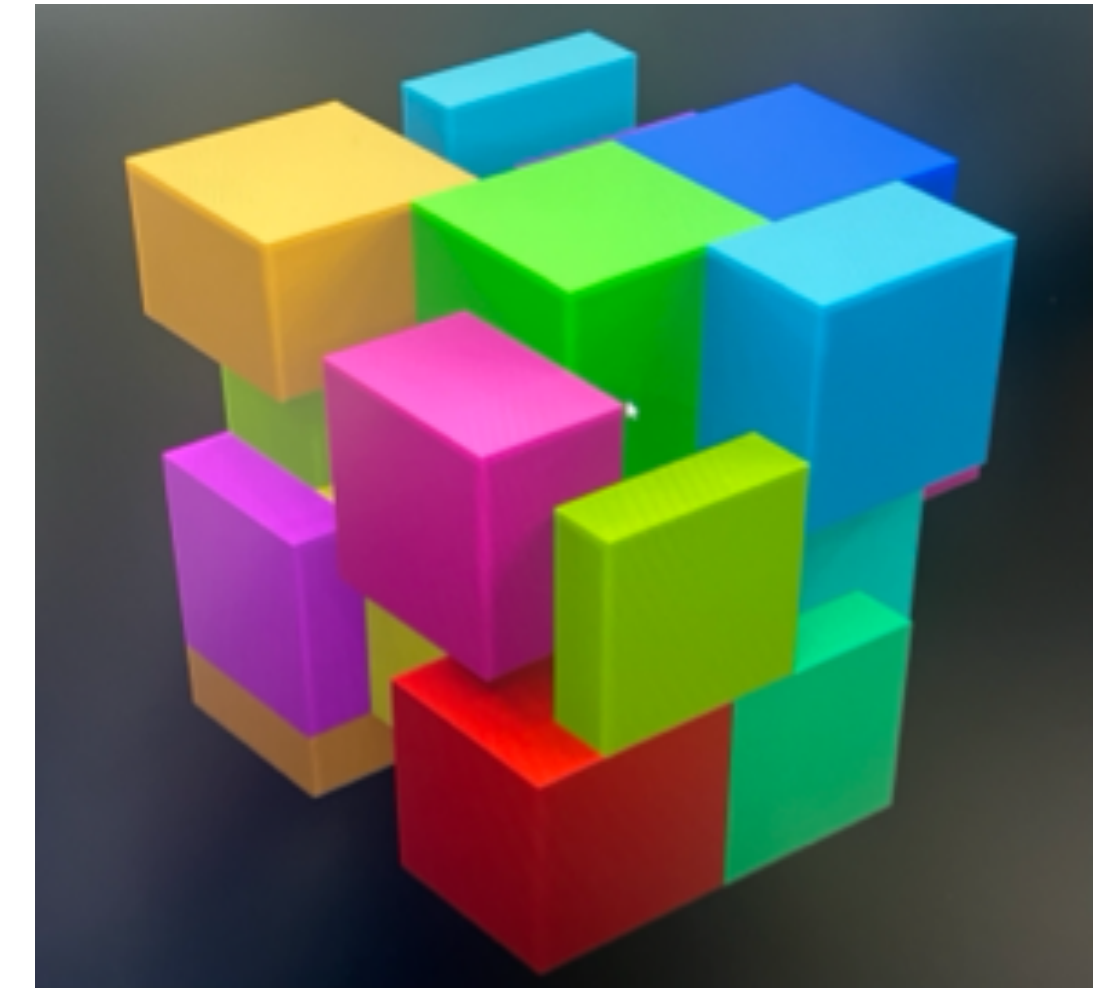
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## Theorem [B., Polak, Zuiddam; 2025]

We have  $\lim_{r \rightarrow \infty} r - \Theta(r) = 0$ .



$$r = 5/2$$



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## Open Problem

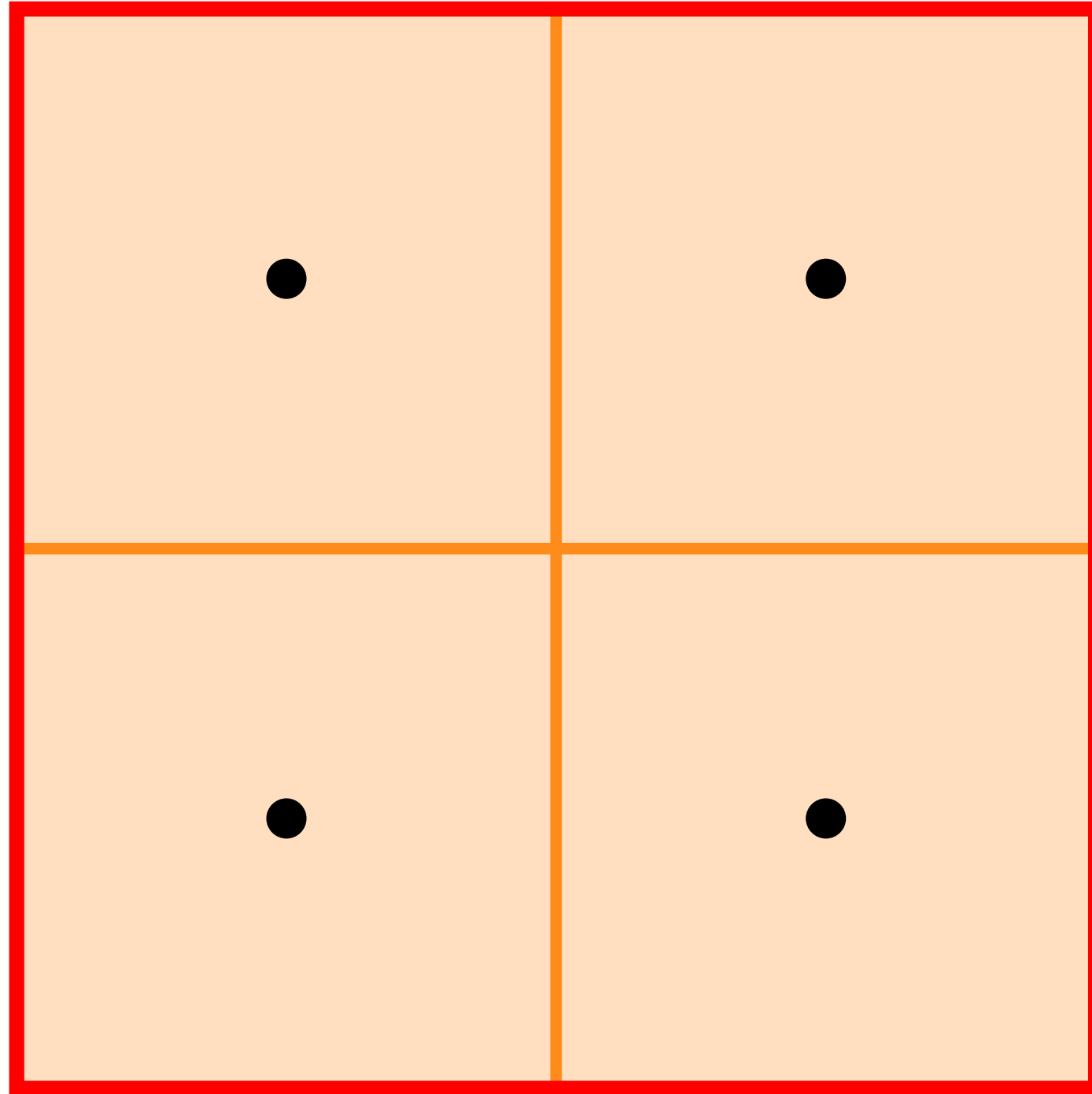
Let  $\Theta_{\text{grp}}(r)$  denote the asymptotic packing density of side length  $r$  torus with lattices.

Do we have  $\Theta_{\text{grp}}(r) = \Theta(r)$ ?

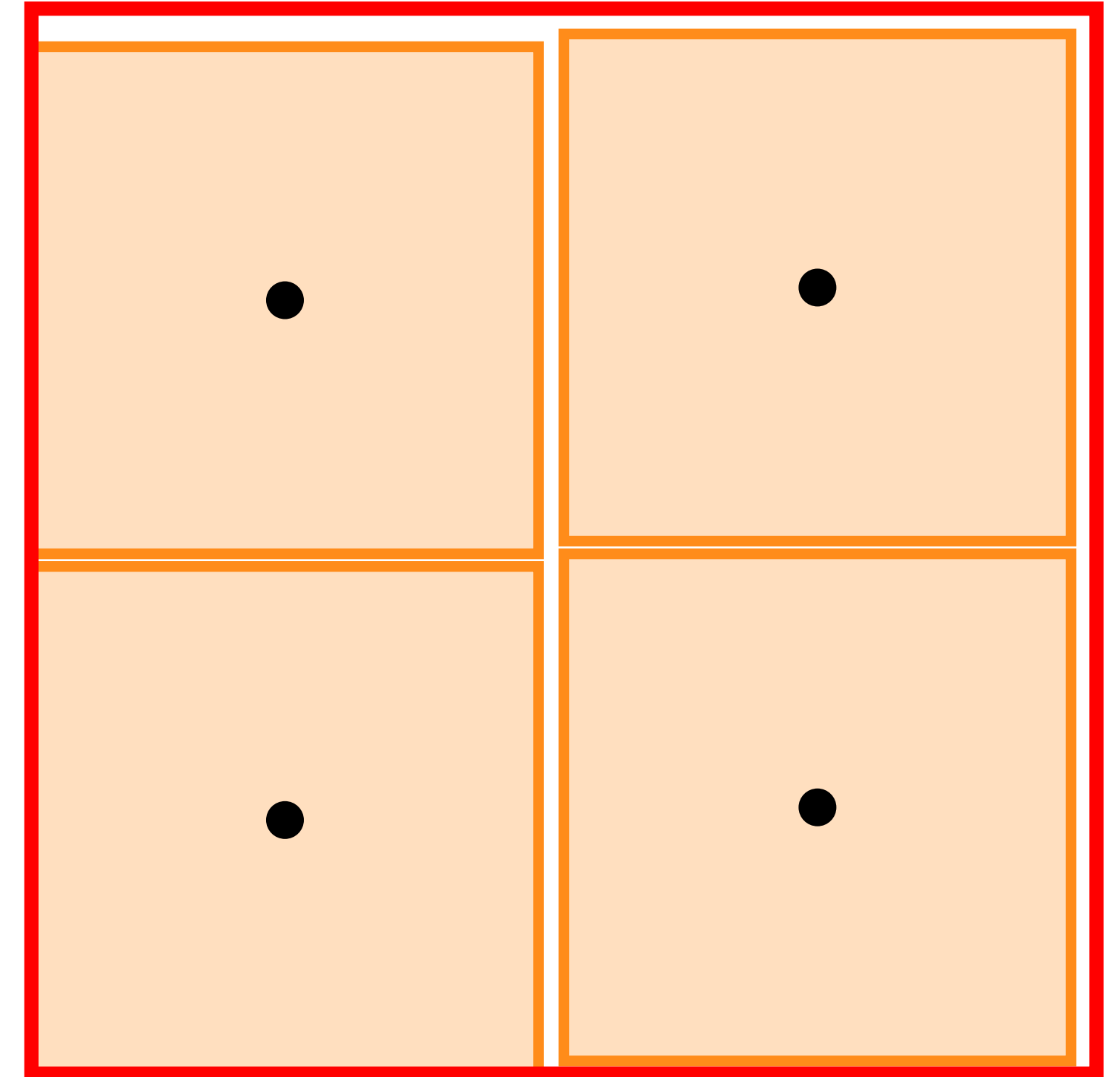
# Theorem [Bohman and Holzman; 2003]

Whenever  $r > 2$  then  $\Theta(r) > 2$ .

$r = 2$



$r > 2$



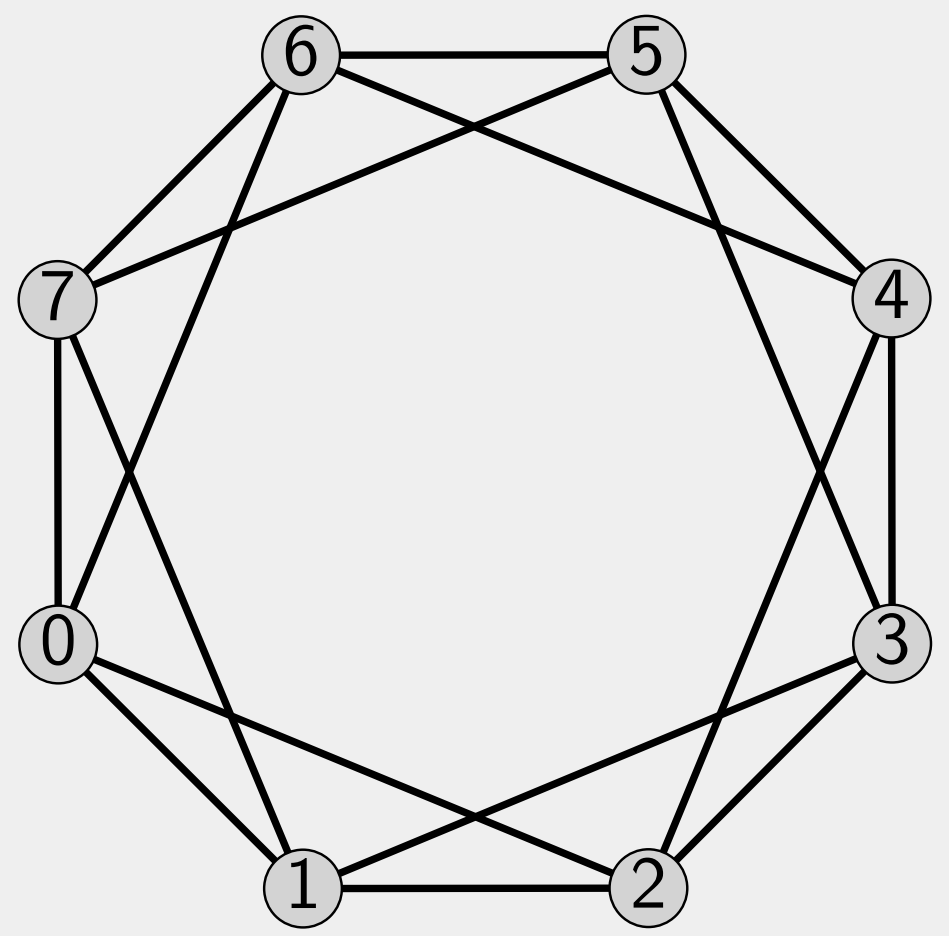
Q. Is  $\Theta$  strictly increasing? I.e. if  $r_1 < r_2$  then  $\Theta(r_1) < \Theta(r_2)$ ?

Q. What about  $\Theta_{\text{grp}}$ ?

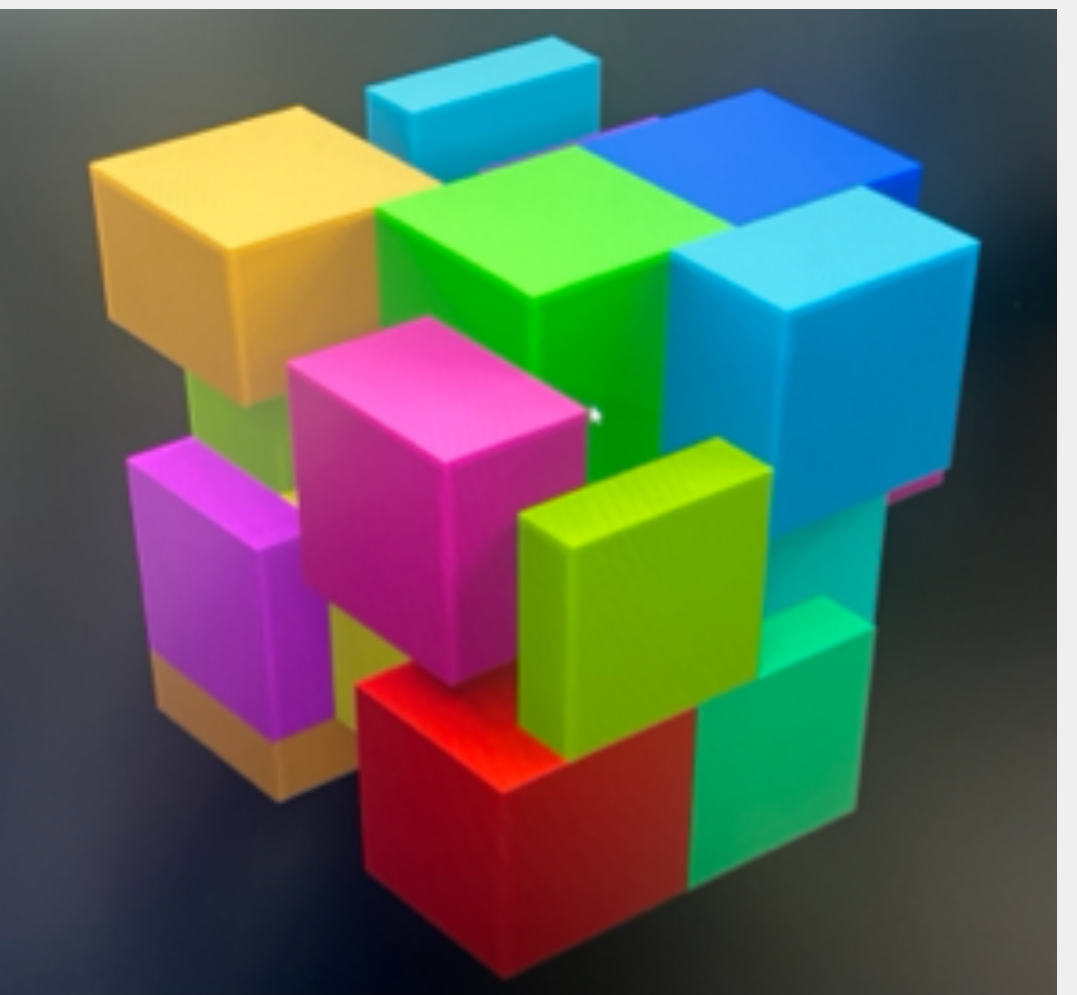
For  $2 \leq p/q$  define the graph  $E_{p/q}$  with

- Vertex set:  $\mathbb{Z}_p$
- Edge set:  $x \sim y$  iff  $-q < (x - y) \pmod p < q$ .

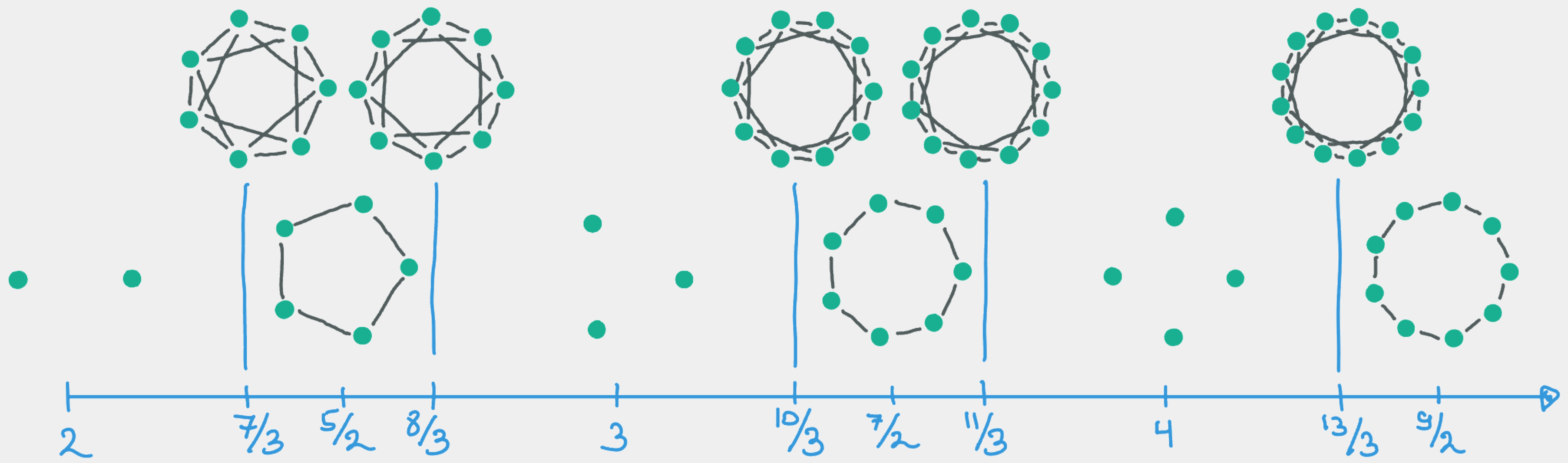
**Lemma:** We have  $\alpha_n(p/q) = \alpha(E_{p/q}^{\boxtimes n})$ .



$E_{8/3}$

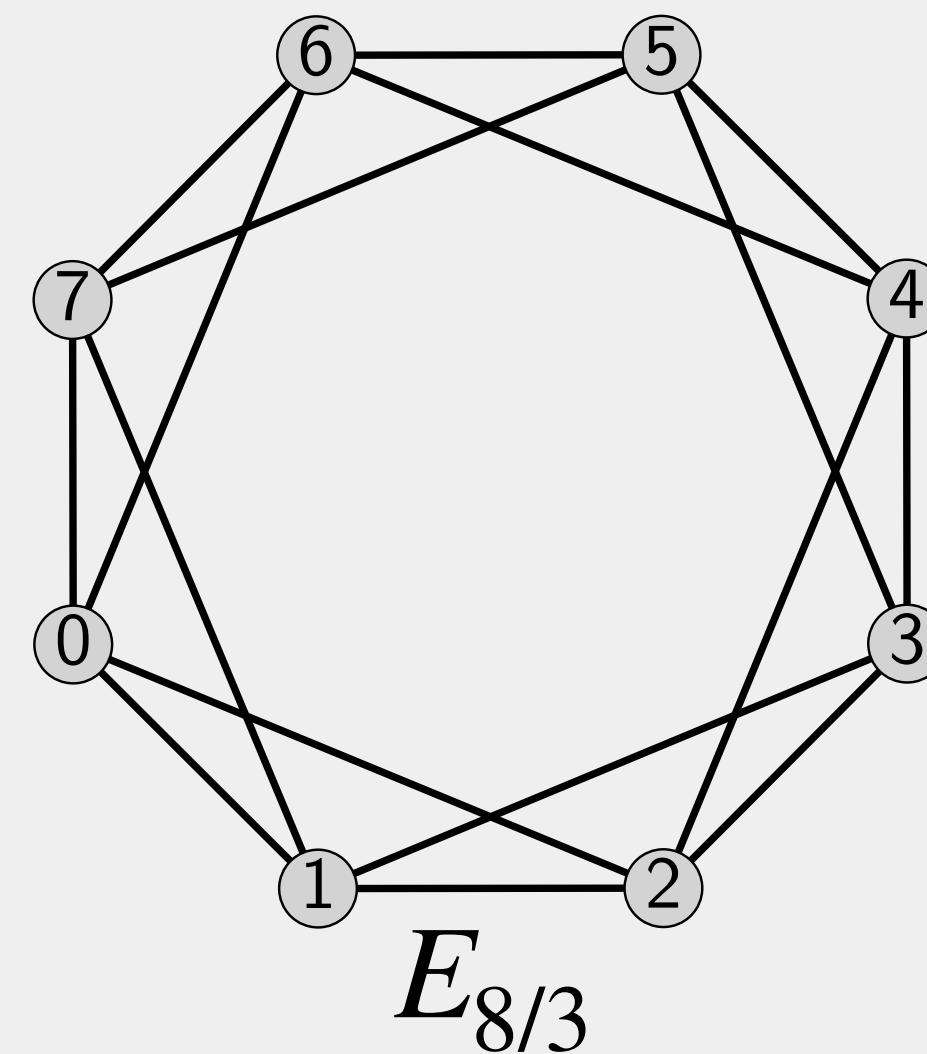


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$$r = 8/3$$

**Lemma:** We have  $\alpha_n(p/q) = \alpha(E_{p/q}^{\boxtimes n})$ .

Every current best lower bound of the Shannon Capacity of odd cycles is **derived** from a lattice in some **nearby**  $E_{p/q}$ .

$G$	$H$	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1, 2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1, 7, 7^2, 7^3, 7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1, 0, 2) + t \cdot (0, 1, 4) : s, t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR <sup>+</sup> 71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1, 11, 11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR <sup>+</sup> 71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1, 19, 117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR <sup>+</sup> 71] <sup>18</sup>
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1, 15, 1073, 1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

# A limit theorem for Shannon capacity from lattice packings

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