Reconfiguration of Independent Transversals

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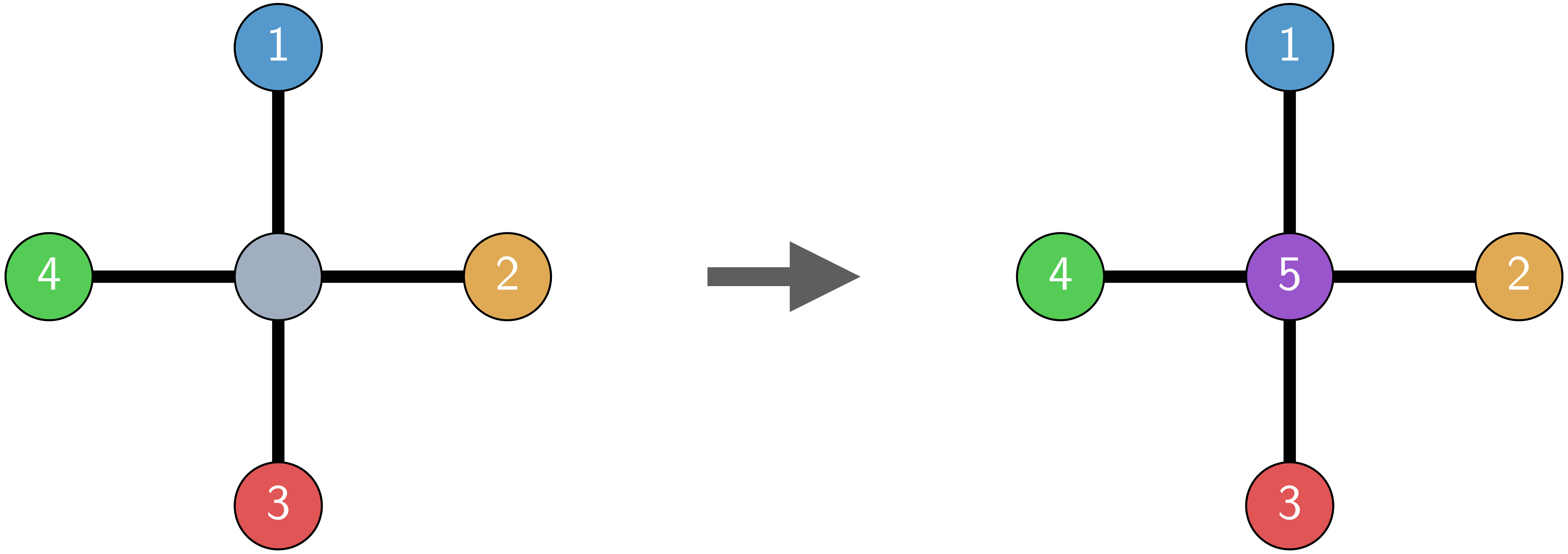
Fact

For any graph G we have $\chi(G) \leq \Delta(G) + 1$.



Question

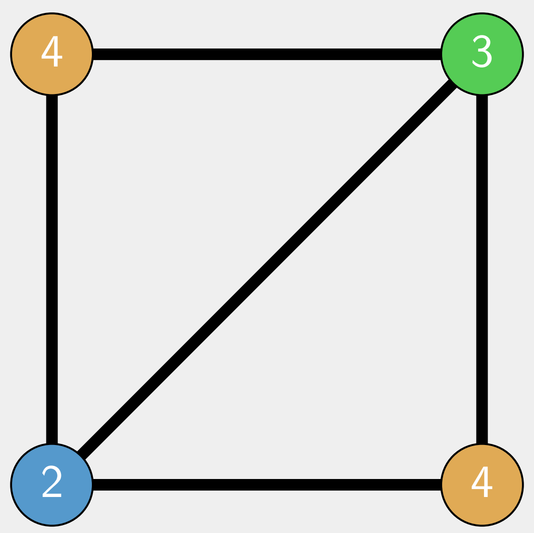
For $q \geq \Delta + 1$, how can we sample a random proper q-coloring uniformly at random?

Input: A graph G and an integer $q \geq \Delta(G) + 1$.

Output: A proper q-coloring generated (close to) uniformly at random.

Glauber Dynamics

We generate a list of random proper q-colorings $X_0, X_1, X_2, \ldots$



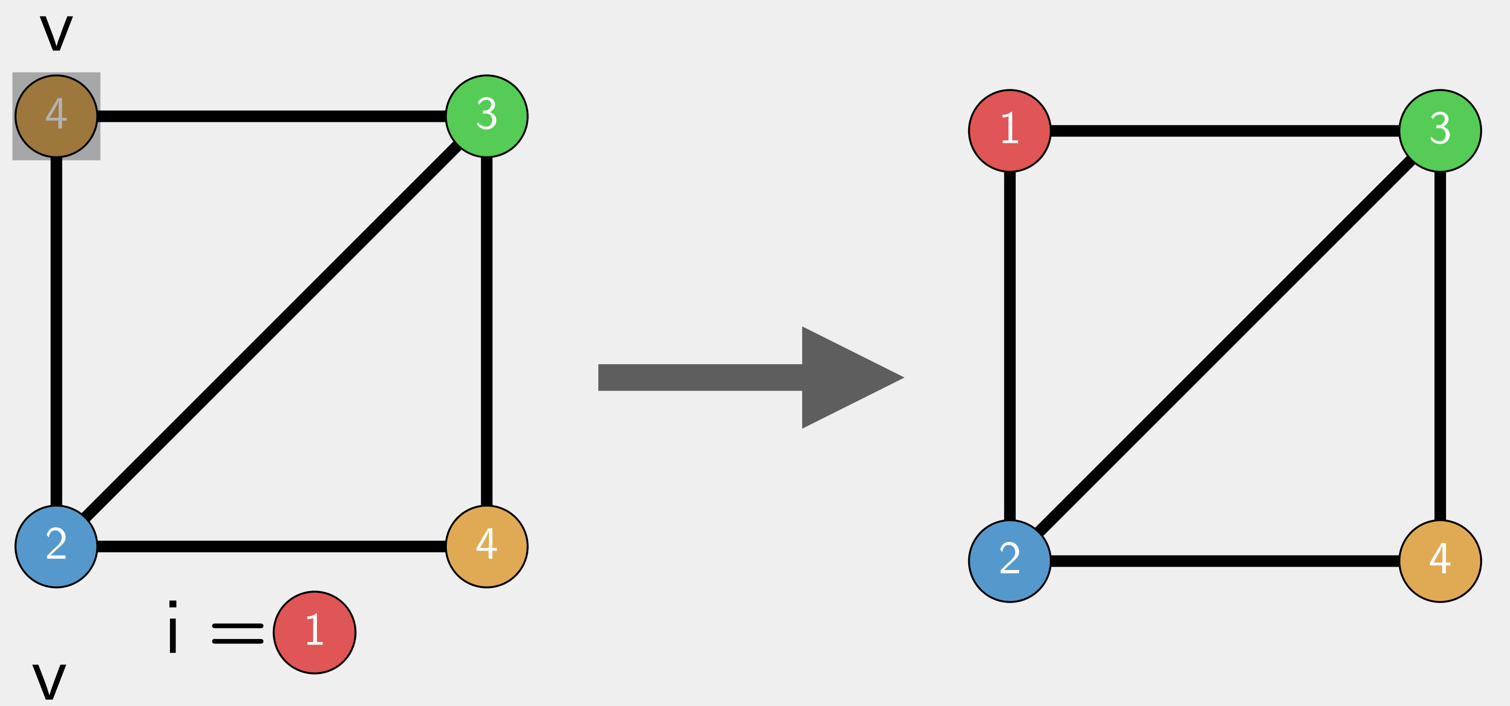
(Randomly) generate a proper q-coloring X_0.

For $t = 1, \ldots, N$:

Uniformly at random pick vertex v and color $i \in [q]$.

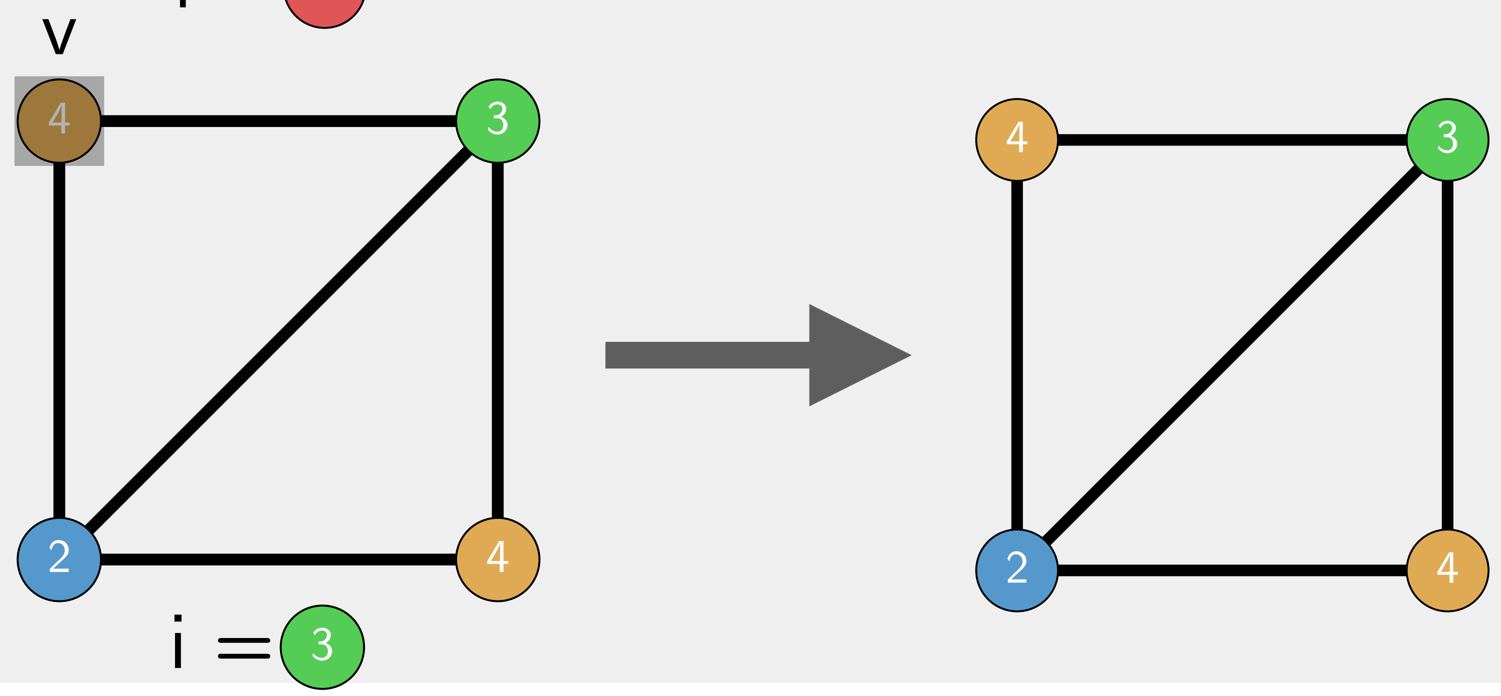
If v in X_{t-1} can be recolored to i:

Let X_t be this recoloring.



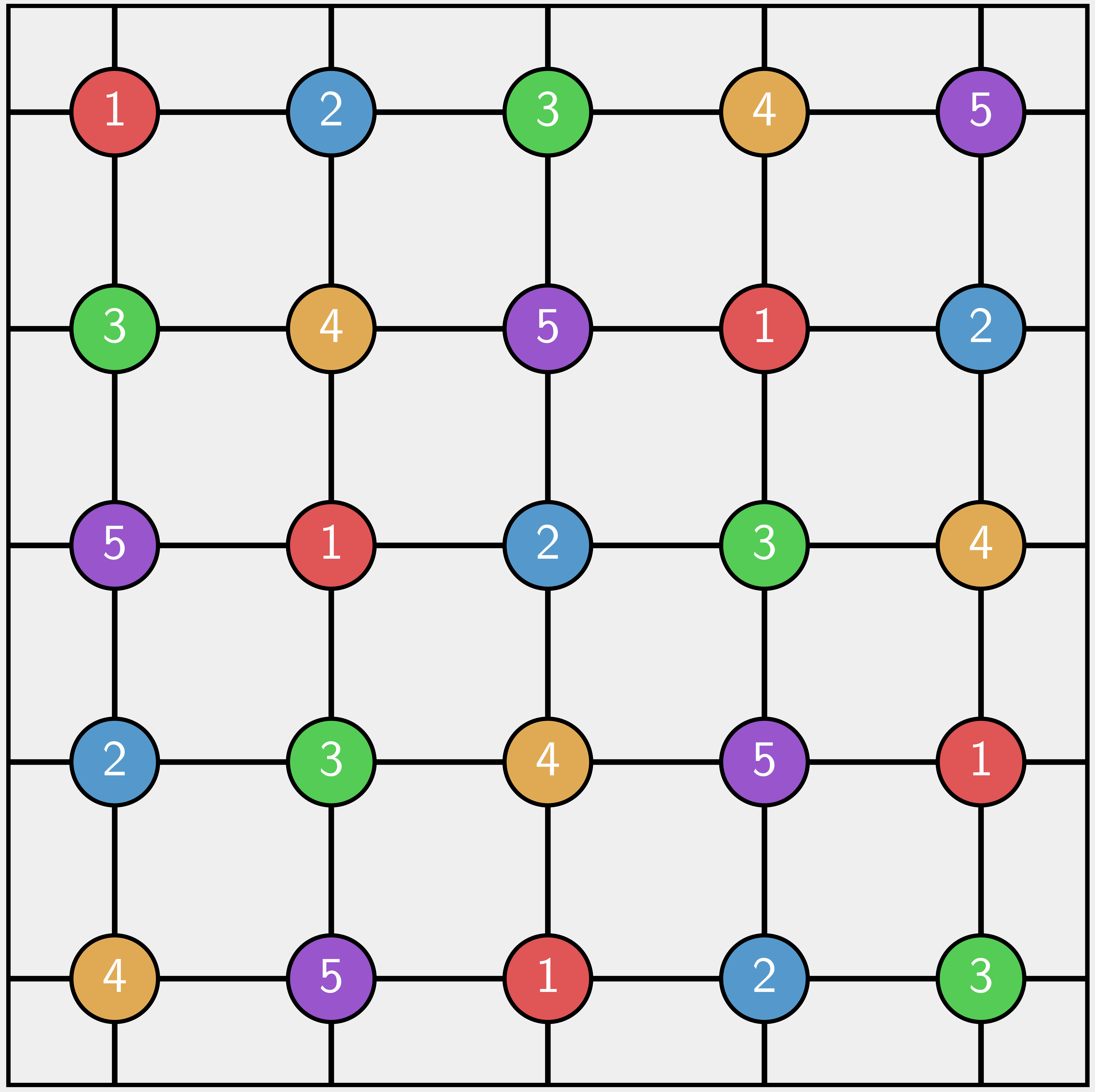
Else:

Let $X_t = X_{t-1}$.



Observations on Glauber Dynamics

- It is a Markov chain.
- It is symmetric, i.e. $\mathbb{P}(\pi_1, \pi_2) = \mathbb{P}(\pi_2, \pi_1)$ for all colorings π_1, π_2.
- It is aperiodic because $\mathbb{P}(\pi, \pi) > 0$.
- It is irreducible if $q \geq \Delta + 2$.



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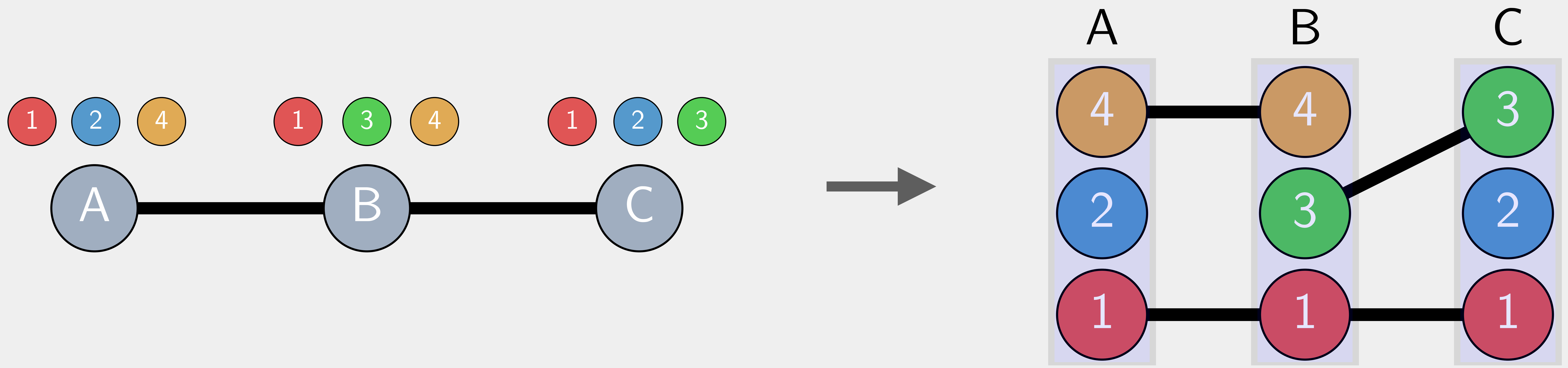
It follows that the Markov chain is ergodic and that X_t converges to the uniform stationary distribution.

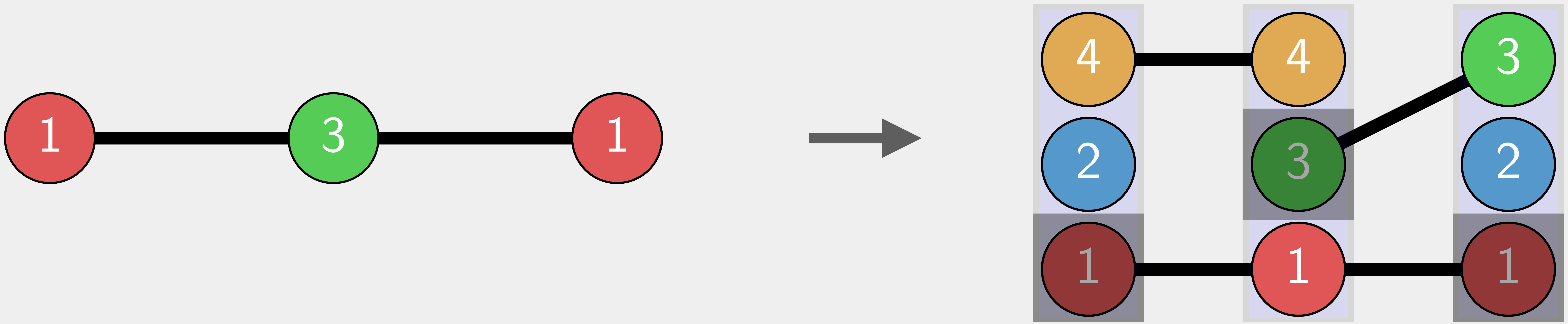
* Lubin, Sokal (1993)

List-Coloring to Independent Transversals

Given a graph H and list assignments $L : V(H) \to \mathscr{P}(\mathbb{N})$ we define $G_{H,L}$ as follows.



Proper L-colorings of H correspond to independent transversals of $G_{H,L}$.



An independent transversal of a pair $(G, \mathscr{U})$, consisting of a graph G and a partition $\mathscr{U}$ of $V(G)$, is an independent set S of G for which $|S \cap U| = 1$ for all $U \in \mathscr{U}$.

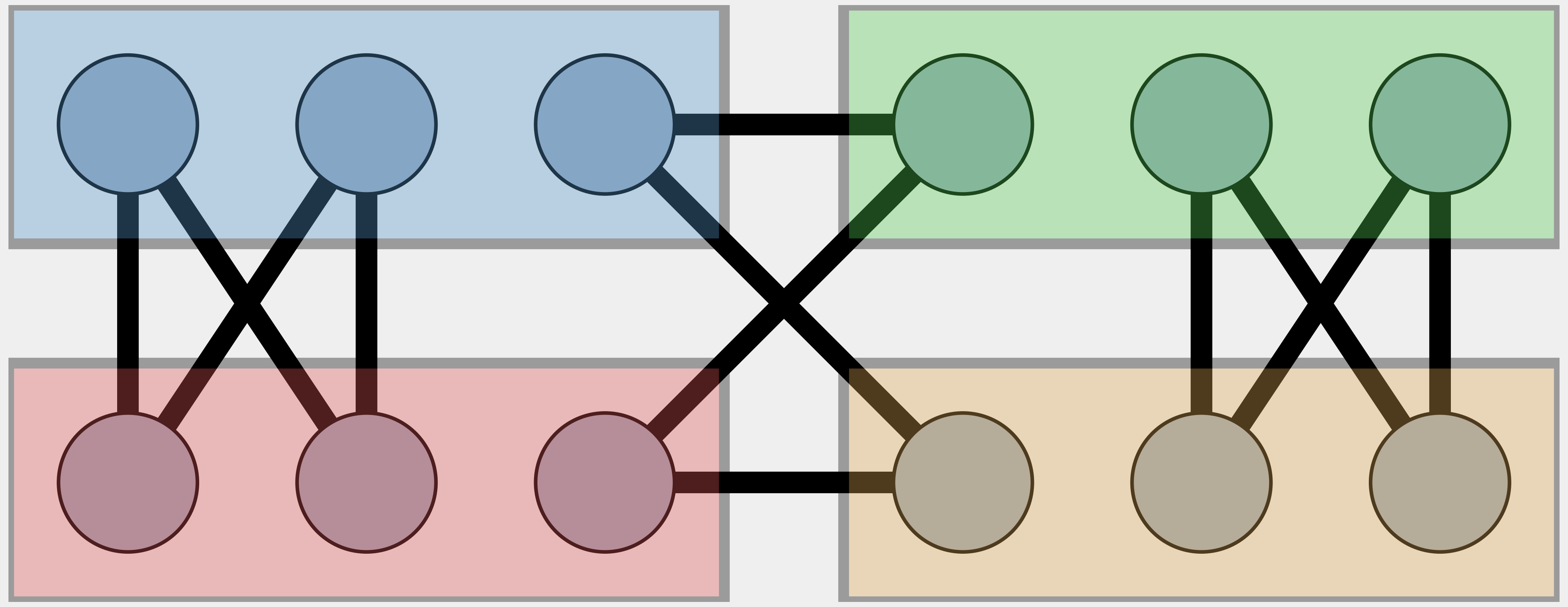
The partition $\mathscr{U}$ is said to be *t*-thick if $|U| \geq t$ for all $U \in \mathscr{U}$.

Theorem (Haxell; 1995)

Let G be a graph of max degree Δ and $\mathscr{U}$ a 2Δ-thick vertex partition. Then there exists an independent transversal $(G, \mathscr{U})$.

Theorem (Szabó, Tardos; 2006)

For any $\Delta \geq 1$ there exists a graph G of max degree Δ with a $(2\Delta - 1)$-thick vertex partition $\mathscr{U}$ such that $(G, \mathscr{U})$ has no independent transversals.

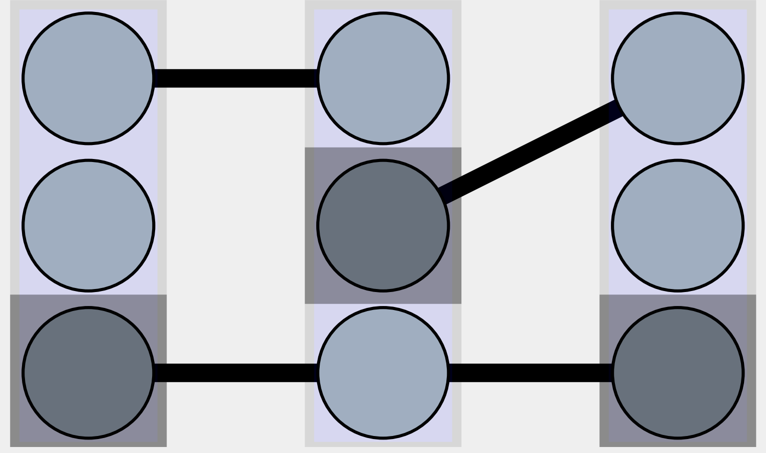


Input: A graph G with $2\Delta(G)$-thick vertex partition $\mathscr{U}$.

Output: An independent transversal of $(G, \mathscr{U})$ generated (close to) uniformly random.

Glauber Dynamics

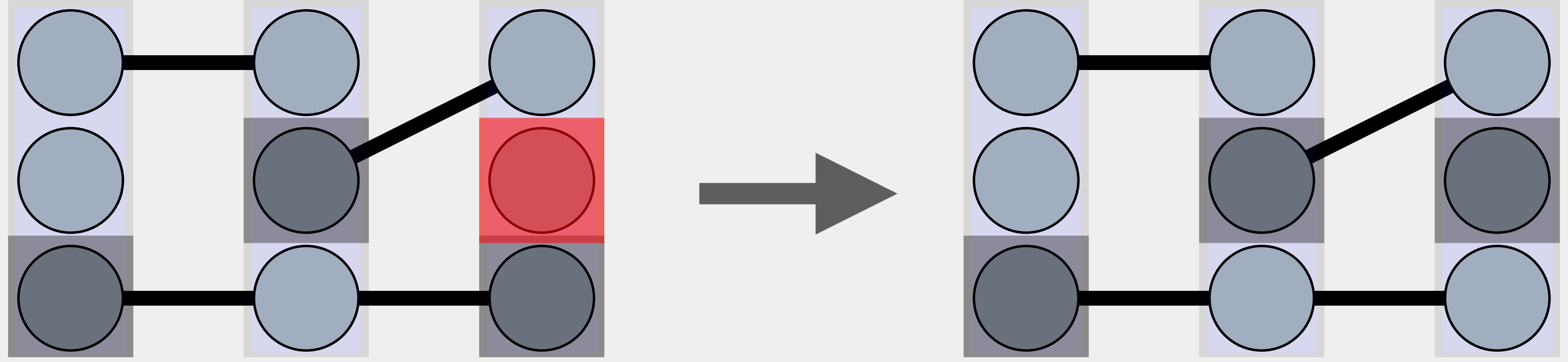
We generate a list of random independent transversals $S_0, S_1, S_2, \ldots$



(Randomly) generate an independent transversal S_0.

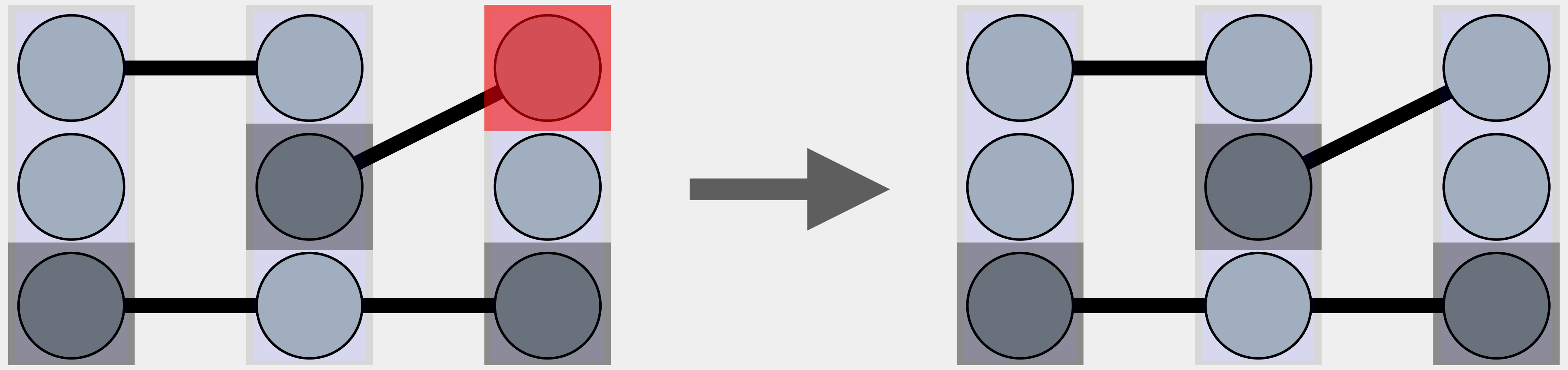
For $t = 1, \ldots, N$:

Uniformly at random pick vertex v.



If v is independent of S_{t-1}:

Let S_t be S_{t-1} with v replaced.

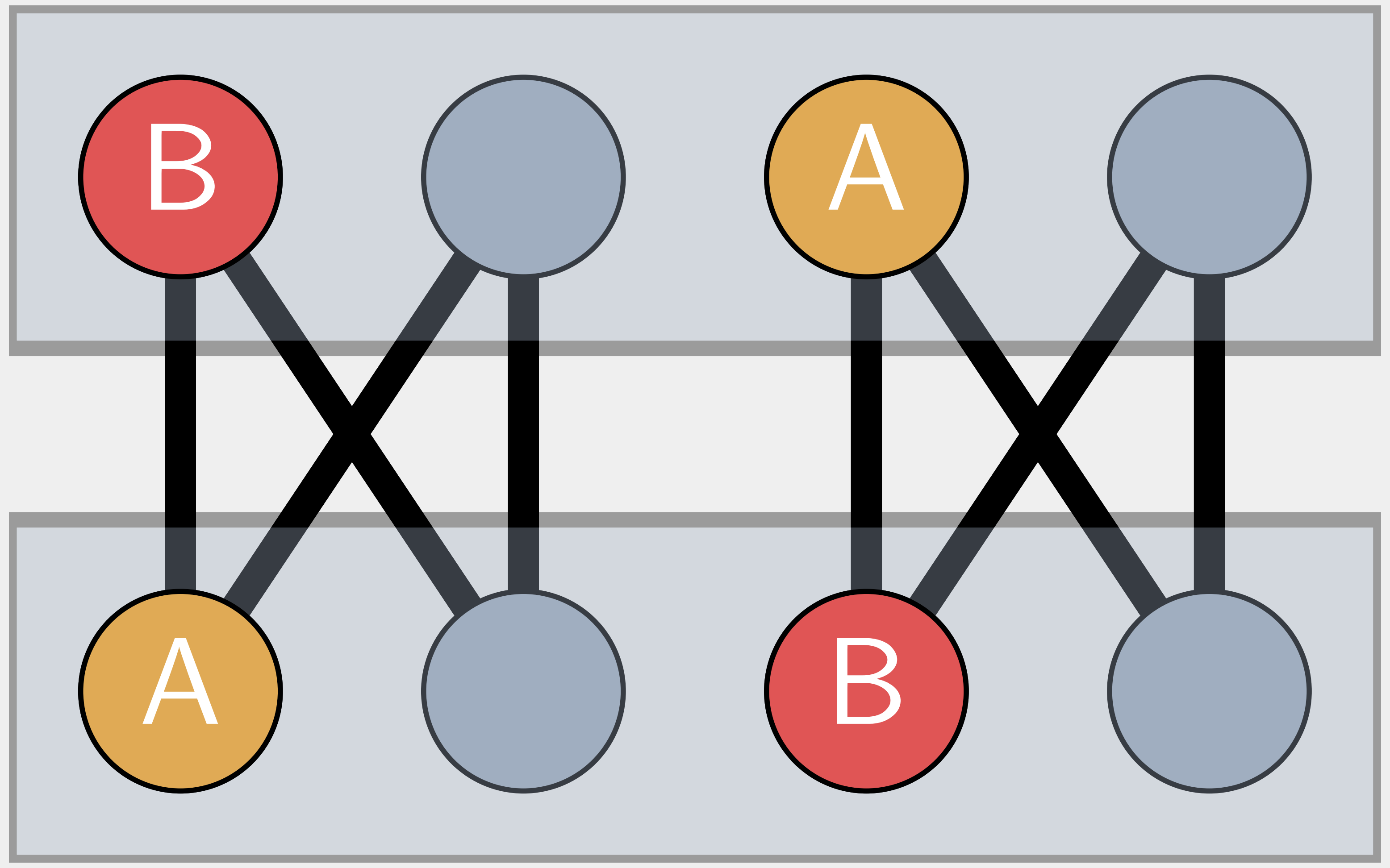


Else:

Let $S_t = S_{t-1}$.

Glauber dynamics of independent transversals is symmetric and aperiodic.

It is not necessarily irreducible.



Given a pair $\mathscr{G} = (G, \mathscr{U})$ we define the reconfigurability graph $R_{\mathscr{G}}$ with vertex set the set of independent transversals of $\mathscr{G}$, where $S \sim T$ iff $|S \cap T| = |\mathscr{U}| - 1$.

Theorem (B., Kang, Ozeki; 2024)

Let $\mathscr{G} = (G, \mathscr{U})$ consist of a graph G of max degree Δ and $\mathscr{U}$ a 2Δ-thick vertex partition. If $\mathscr{G}$ is irreducible and G is not isomorphic to the disjoint union of $|\mathscr{U}|$ copies of $K_{\Delta,\Delta}$, then $R_{\mathscr{G}}$ is connected.

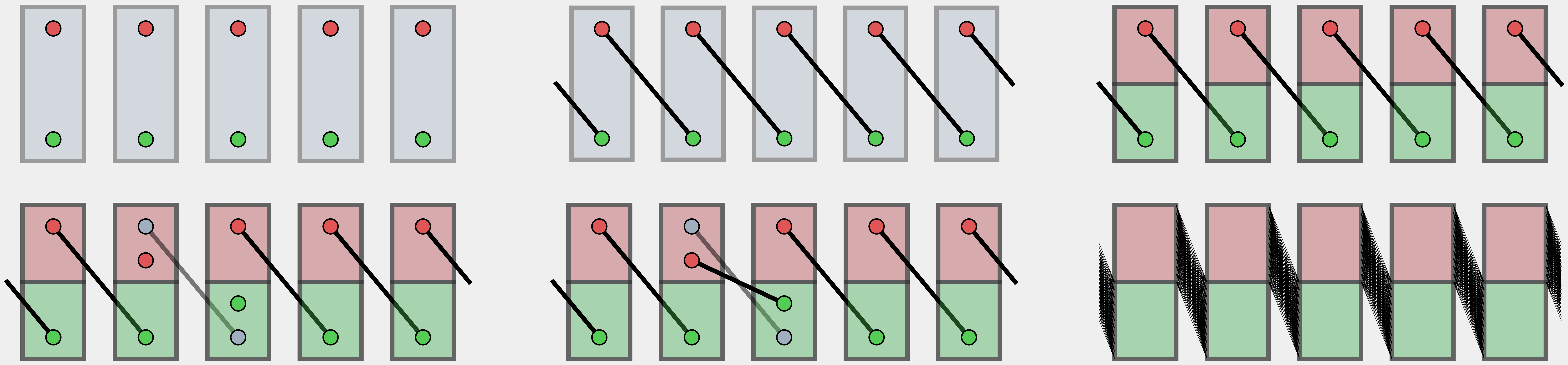
Lemma

Let $\mathscr{G} = (G, \mathscr{U})$ a graph G with 2Δ-thick partition $\mathscr{U}$. Suppose S,T are independent transversals of $\mathscr{G}$ that cannot be reconfigured to intersect, then $G \cong |\mathscr{U}| \cdot K_{\Delta,\Delta}$.

Proof. Every $v \in V(G)$ has exactly one neighbor in $S \cup T$ because:

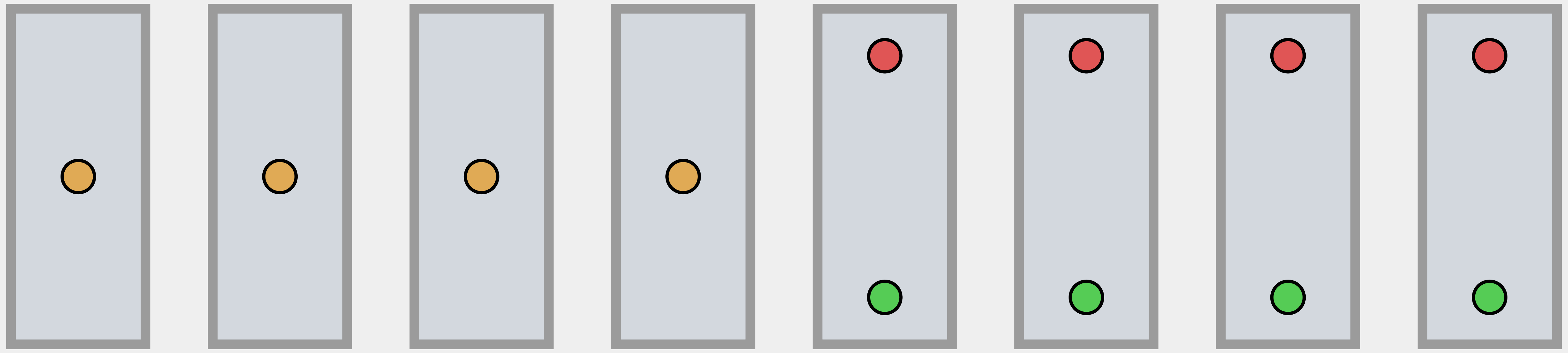
$$2\Delta\,|\mathscr{U}| \leq |V(G)| \leq \sum_{v \in V(G)} |N_G(v) \cap (S \cup T)| = \sum_{u \in S \cup T} |N_G(u)| \leq \Delta \cdot |S \cup T| = 2\Delta\,|\mathscr{U}|\,.$$

So $V(G)$ can be partitioned into $N_G(S)$ and $N_G(T)$. Take a vertex pair from $N_G(S)$ and in $N_G(T)$ in the matched block. Now (S,T) can be reconfigured to include these, so they must be adjacent. This reveals complete bipartite graphs which must all be $K_{\Delta,\Delta}$.

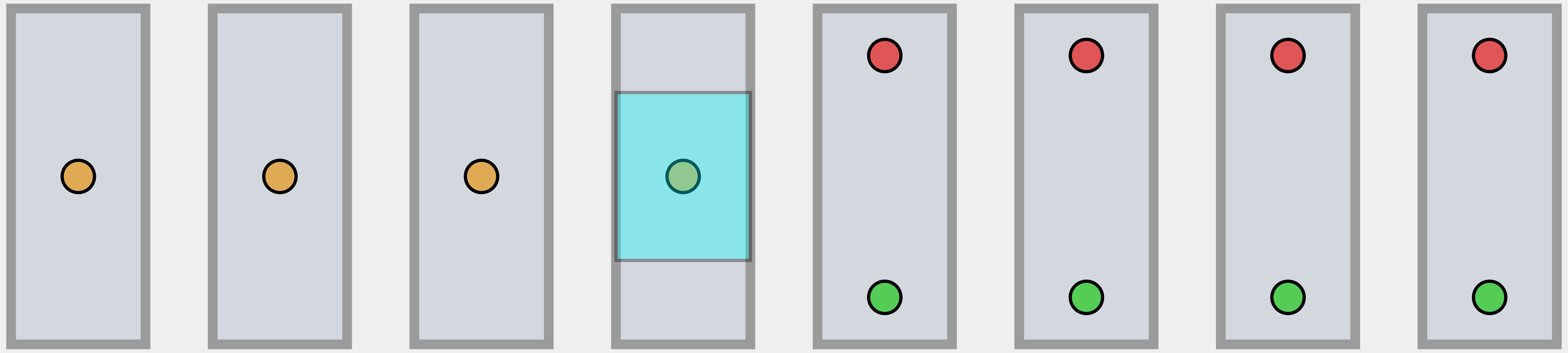


Proof Sketch

Given $\mathscr{G} = (G, \mathscr{U})$ a graph G with 2Δ-thick partition $\mathscr{U}$ with $G \ncong |\mathscr{U}| K_{\Delta,\Delta}$ and two independent transversals S,T, we reconfigure them to be maximally close to each other.



If unequal a vertex of $S \cap T$ is contained in a component $K \cong K_{\Delta,\Delta}$.



We transform $\mathscr{G}$ to $\mathscr{G}' = (G - K, \mathscr{U}')$. By induction we can reconfigure the projections of S and T on $\mathscr{G}'$ which corresponds to a valid reconfiguration on $\mathscr{G}$.

The mixing time of an ergodic Markov chain with P with stationary distribution π is

$$t_{\mathsf{M}} = \min\left\{t \in \mathbb{N} : \max_{\nu} d_{\mathsf{TV}}\left(\pi, \nu P^t\right) \leq \frac{1}{4}\right\}.$$

Glauber dynamics is said to mix rapidly if $t_{\mathsf{M}} = \mathcal{O}(\text{poly(n)})$, where $n = |V(G)|$.

Glauber dynamics for coloring

Glauber dynamics was shown to mix rapidly whenever $q \geq 2\Delta + 1$ [Jerrum; 1995], improved to $q \geq \frac{11}{6}\Delta$ [Vigoda; 1999], and $q \geq (1-\epsilon)\frac{11}{6}\Delta$ [Chen et al.; 2019].

It has been conjectured to mix rapidly whenever $q \geq \Delta + 2$.

Glauber dynamics for independent transversals

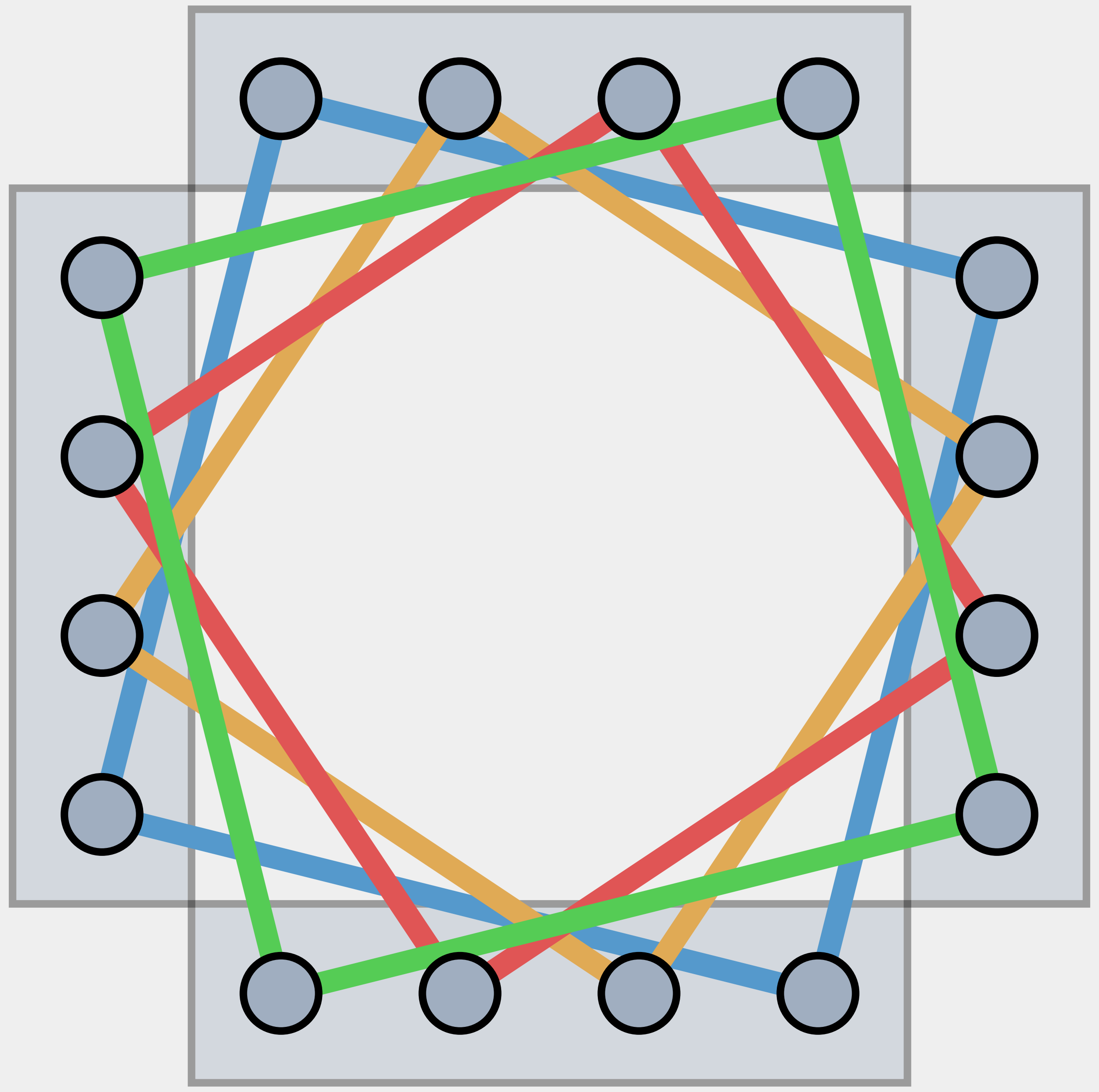
Question. Is there any constant $c > 2$ such that Glauber dynamics of ITs mixes rapidly on pairs $(G, \mathcal{U})$ where $\mathcal{U}$ is $(c\Delta)$-thick?

Question - Characterization

Our theorem. For irreducible $\mathscr{G} = (G, \mathscr{U})$, graph G with $\mathscr{U}$ a 2Δ-thick vertex-partition:

$$G \ncong |\mathscr{U}| \cdot K_{\Delta,\Delta} \;\Rightarrow\; R_{\mathscr{G}} \text{ connected}\,.$$

The converse does not hold.

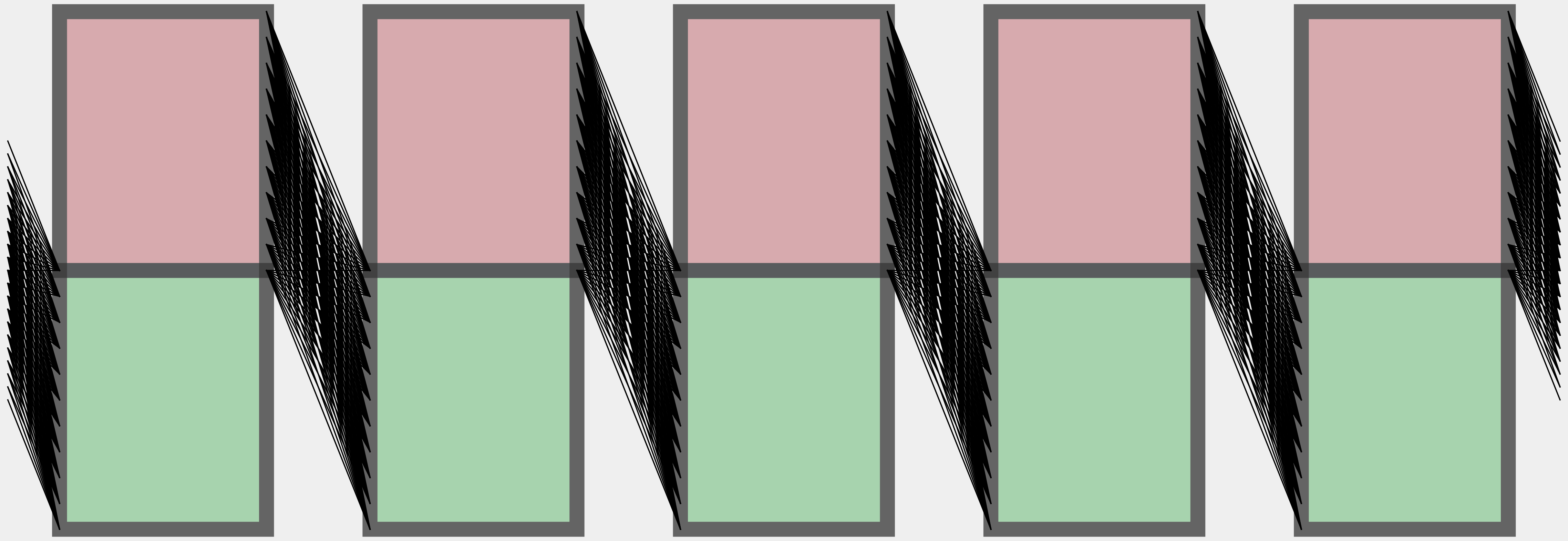


Question. Can we characterize exactly those $\mathscr{G}$ for which $R_{\mathscr{G}}$ is disconnected?

Question - Extremality

Question. What are the configurations of pairs $(G, \mathscr{U})$ of graphs with 2Δ-thick vertex-partition that have the fewest independent transversals?

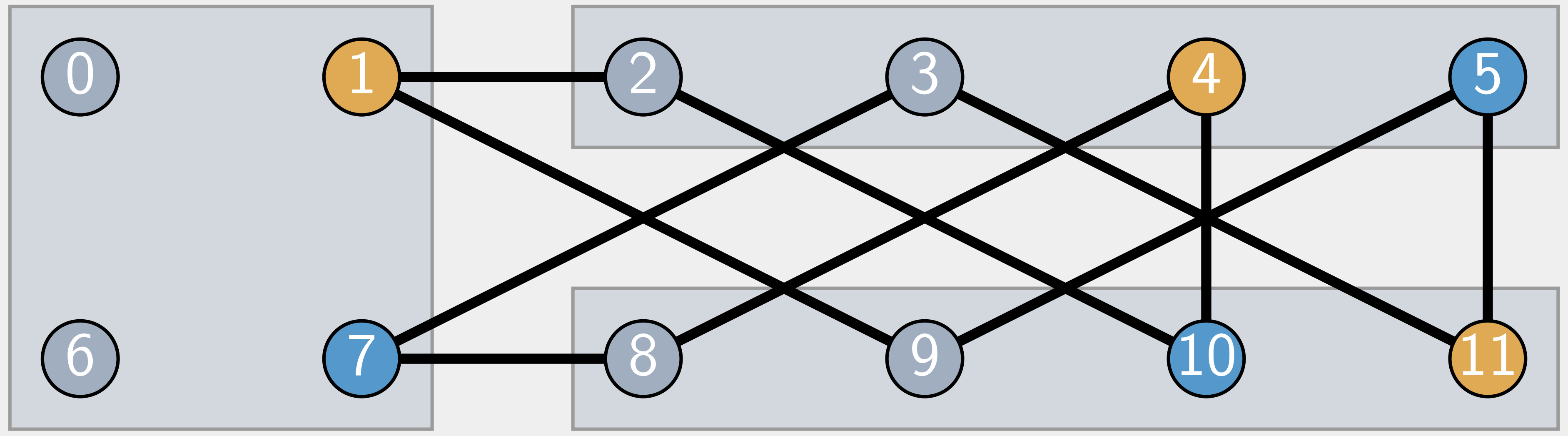
Conjecture. We think the following is a natural candidate for being extremal.

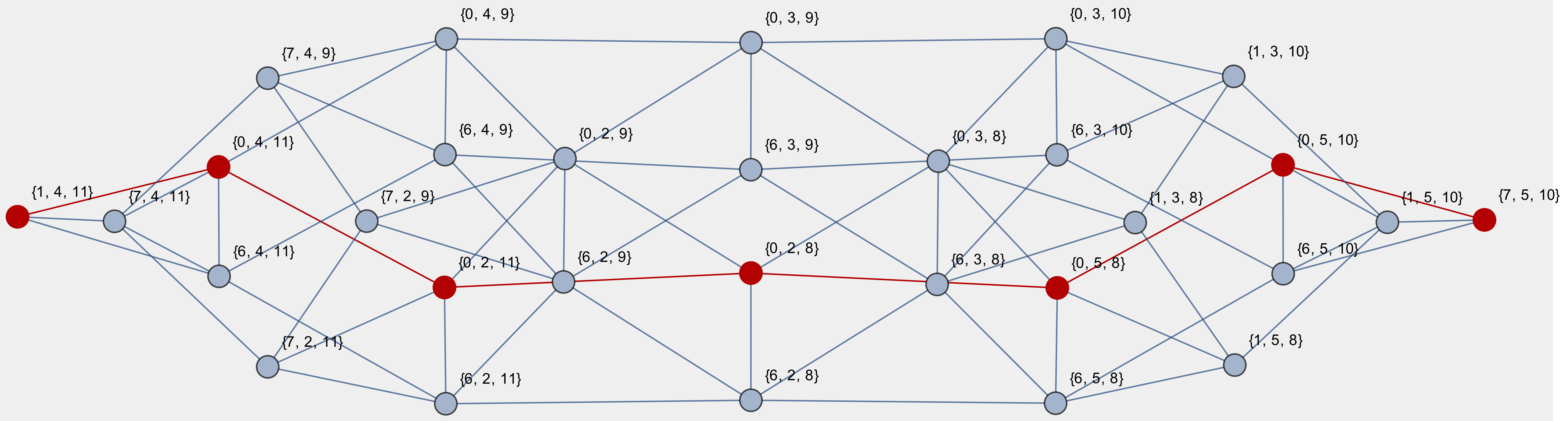


It has $2\Delta^{|\mathscr{U}|}$ distinct independent transversals.

More Questions

Q. Can we give an upper bound for the diameter of $R_{\mathscr{G}}$ as a function of $|\mathscr{U}|$ and Δ?





Q. Haxell's Theorem for the existence of independent transversals for (2Δ)-thick has a topological proof. Can our result be proven in a similar way?

Q. There are several structures between (list)-coloring and independent transversals, e.g. bounded color-degree or bounded local-degree. What about connectedness of $R_{\mathscr{G}}$ for those problems (in the existence regime)?

Q. Does a fast approximate uniform sampler of ITs imply an FPRAS for counting ITs?

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