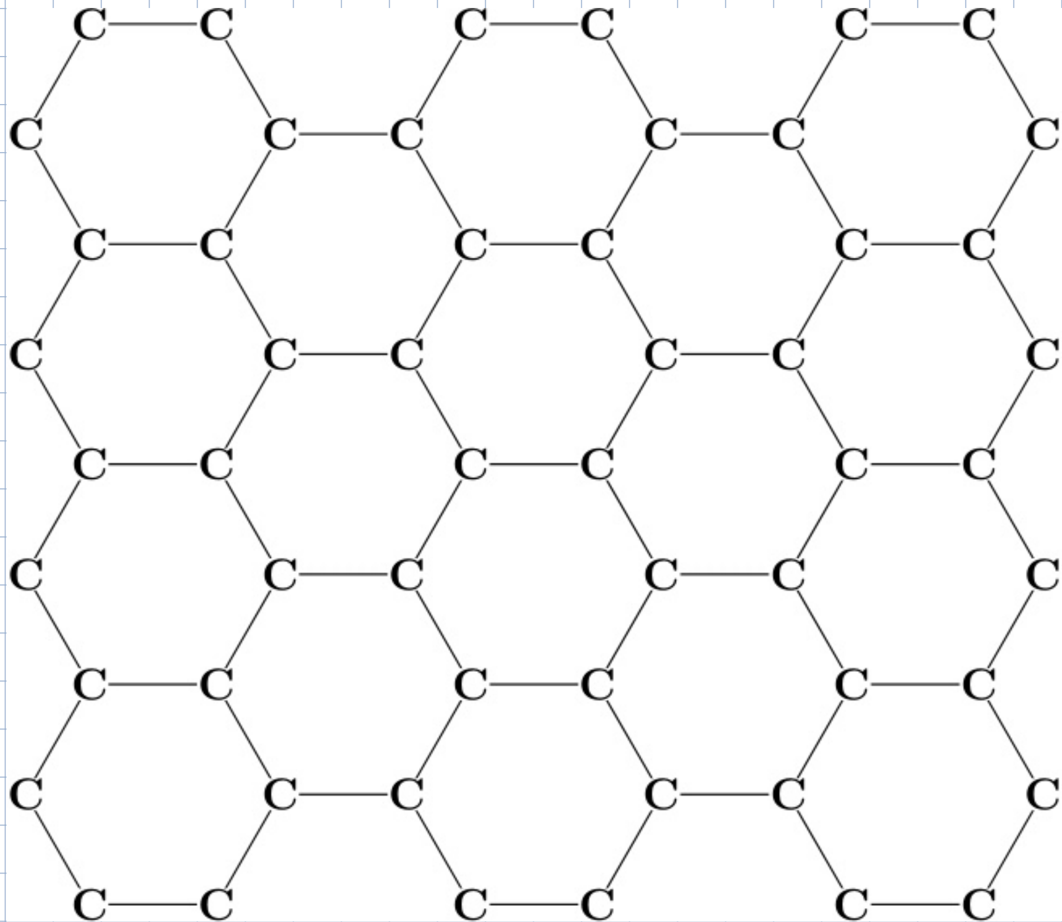
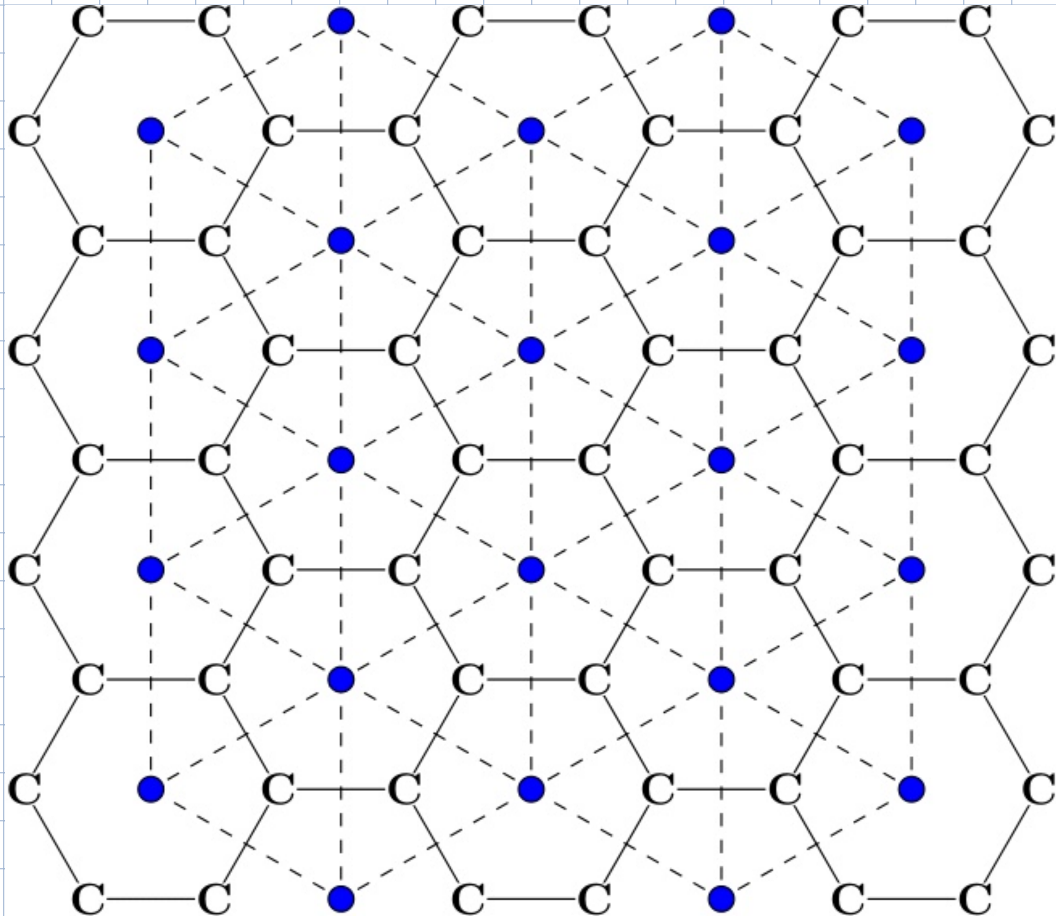
Quasiworld Seminar:

Complex dynamics and the independence polynomial

Pjotr Buys

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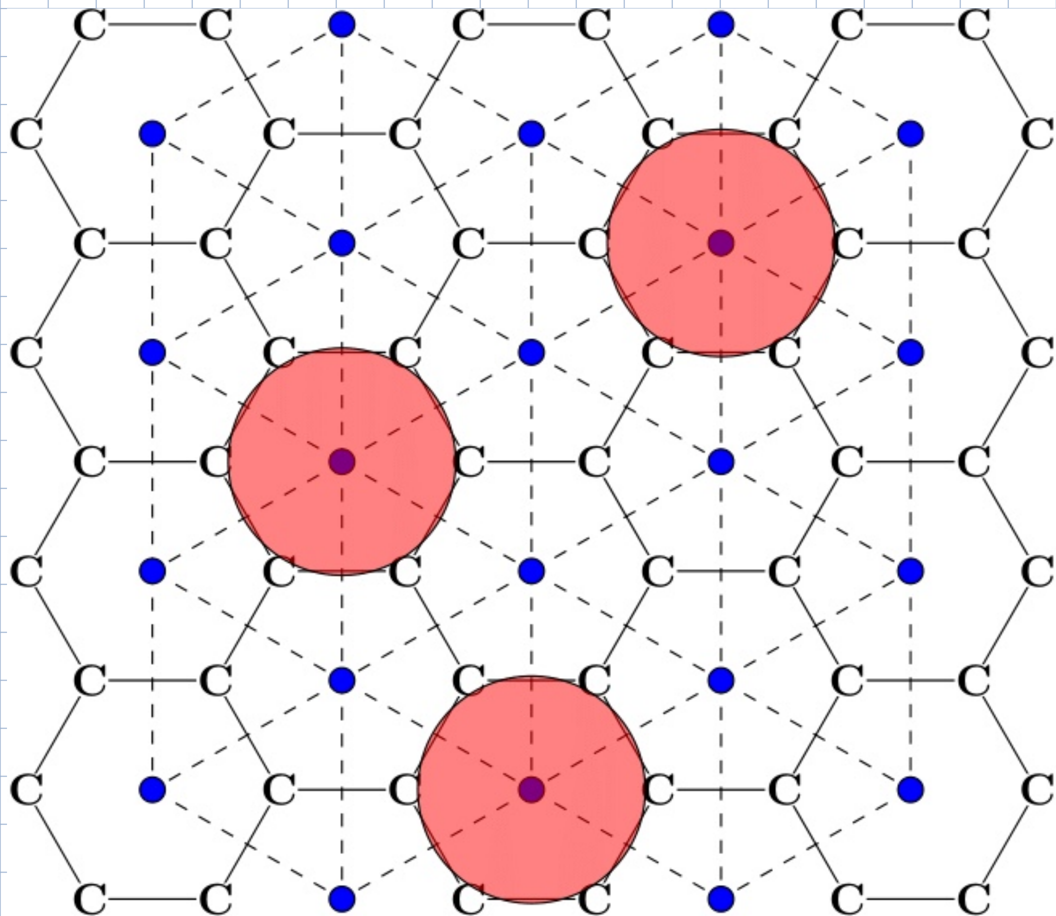
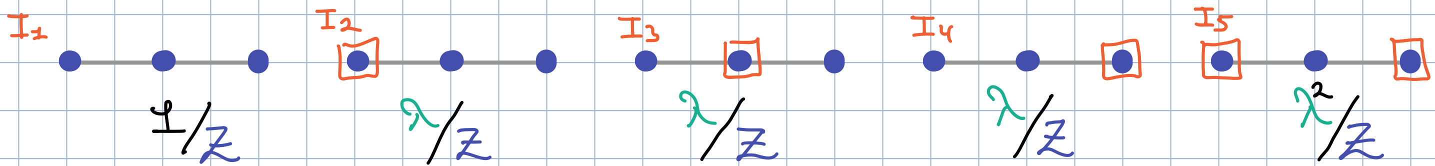


The Hard-Core Model

Given a graph $G=(V,E)$ an independent set is a subset $I \subseteq V$ that does not span any edges.



The probability of observing an independent set I is proportional to $\lambda^{|I|}$, where $\lambda \geq 0$ is called the fugacity parameter.

So the actual probability of observing I is $\lambda^{|I|}/Z$, where

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V(G) \\ \text{independent}}} \lambda^{|I|}.$$

This is called the partition function of the hard-core model or the independence polynomial.

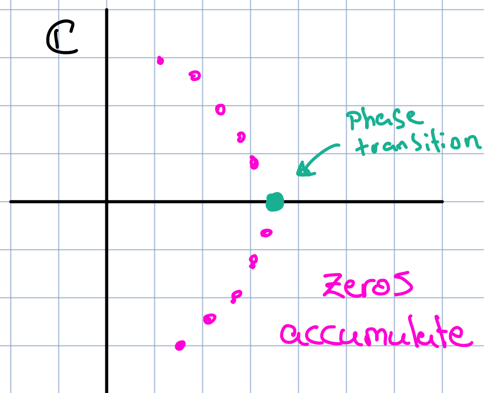
The Complex Zeros of Z_G

Theorem [Lee-Yang 1952]:

Let $G_1, G_2, G_3, \ldots$ be a sequence of graphs converging to a lattice $\mathcal{L}$. Suppose there is an open $U \subseteq \mathbb{C}$ containing $\lambda \geq 0$ such that Z_{G_m} is non-zero on U for all m. Then the pressure of $\mathcal{L}$ defined as

$$P_{\mathcal{L}}(\lambda) = \lim_{m \to \infty} \frac{1}{|G_m|} \log(Z_{G_m}(\lambda))$$

is analytic at λ.



The Complex Zeros of Z_G

For $\Delta \in \mathbb{Z}_{\geq 2}$ we let $\mathcal{G}_\Delta$ denote the set of graphs with max degree $\leq \Delta$.

Theorem [Barvinok 2016, Patel-Regts 2017]:
Let U be a complex domain containing zero such that Z_G is non-zero on U for all $G \in \mathcal{G}_\Delta$. Then for all $\lambda \in U$ there is a "fast" algorithm for approximating $Z_G(\lambda)$ for $G \in \mathcal{G}_\Delta$.

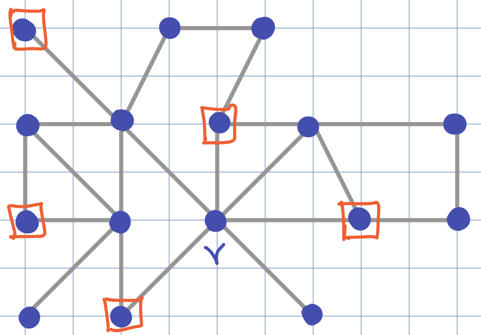
We define the zero-locus $Z_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}$.

Theorem [de Boer, B., Guerini, Peters, Regts 2021]:
Let $\lambda \in Z_\Delta$. Arbitrarily close to λ there are parameters λ_0 for which approximating $Z_G(\lambda_0)$ is computationally hard for $G \in \mathcal{G}_\Delta$. (#P-Hard)

Ratios

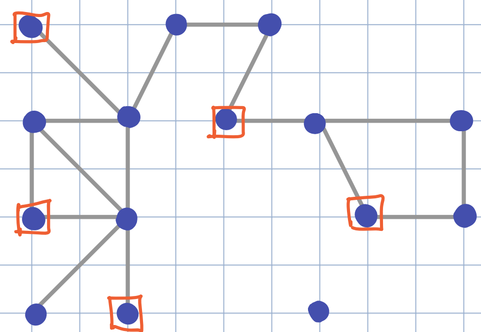
Let $G=(V,E)$ be a graph and let $v \in V$. We split Z_G up:

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V(G) \\ \text{independent}}} \lambda^{|I|} = \underbrace{\sum_{\substack{I \subseteq V(G) \\ \text{independent} \\ v \notin I}} \lambda^{|I|}}_{Z^{out}_{G,v}(\lambda)} + \underbrace{\sum_{\substack{I \subseteq V(G) \\ \text{independent} \\ v \in I}} \lambda^{|I|}}_{Z^{in}_{G,v}(\lambda)}$$



I in G with $v \notin I$

$\longleftrightarrow$



I in $G \setminus \{v\}$

So $Z^{out}_{G,v}(\lambda) = Z_{G-v}(\lambda)$

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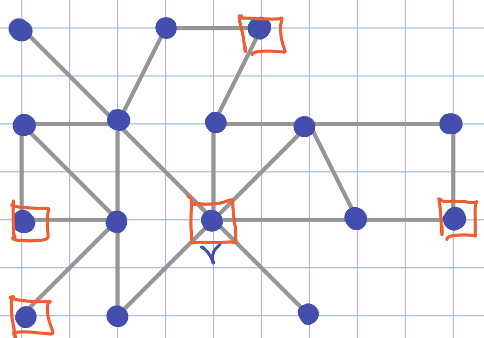
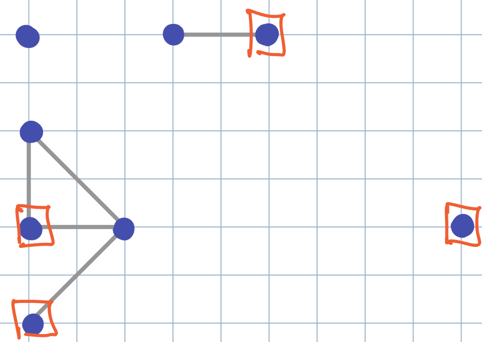


I in G with $v\in I$



I in $G\setminus N[v]$

So $Z^{in}_{G,v}(\lambda) = \lambda\cdot Z_{G-N[v]}(\lambda)$.

Ratios

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Lemma: We have $Z^{out}_{G,v}(\lambda)=Z_{G-v}(\lambda)$ and $Z^{in}_{G,v}(\lambda)=\lambda\cdot Z_{G-N[v]}(\lambda)$.

We define the Ratio as $R_{G,v}(\lambda)=Z^{in}_{G,v}(\lambda)/Z^{out}_{G,v}(\lambda)$ and thus $R_{G,v}$ is a rational function $\hat{\mathbb{C}}\to\hat{\mathbb{C}}$. We let $\mathcal{R}_\Delta=\{R_{G,v}: G\in\mathcal{G}_\Delta \text{ and } v\in V(G)\}$.

We define the activity-locus:

$$A_\Delta=\{\lambda\in\mathbb{C}: \text{there is no neighborhood of } \lambda \text{ on which } \mathcal{R}_\Delta \text{ is a normal family}\}.$$

Theorem: The closure of the zero-locus equals the activity-locus, i.e. $\overline{Z_\Delta}=A_\Delta$.

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Proof $\overline{Z_\Delta} \supseteq A_\Delta$: Take some $\lambda_0 \notin \overline{Z_\Delta}$ with neighborhood U on which Z_G is non-zero for all $G \in \mathcal{G}_\Delta$.

Claim: $R_{G,v}(\lambda) \notin \{0, \infty, -1\}$ for any $G \in \mathcal{G}_\Delta, v \in V(G), \lambda \in U$.

$$R_{G,v}(\lambda) = 0 \Leftrightarrow Z^{in}_{G,v}(\lambda) / Z^{out}_{G,v}(\lambda) = 0 \Rightarrow Z^{in}_{G,v}(\lambda) = 0 \Leftrightarrow Z_{G-N[v]}(\lambda) = 0 \quad \lightning$$

$$R_{G,v}(\lambda) = \infty \Leftrightarrow Z^{in}_{G,v}(\lambda) / Z^{out}_{G,v}(\lambda) = \infty \Rightarrow Z^{out}_{G,v}(\lambda) = 0 \Leftrightarrow Z_{G-v}(\lambda) = 0. \quad \lightning$$

$$R_{G,v}(\lambda) = -1 \Leftrightarrow Z^{in}_{G,v}(\lambda) / Z^{out}_{G,v}(\lambda) = -1 \Rightarrow Z^{in}_{G,v}(\lambda) = -Z^{out}_{G,v}(\lambda) \Leftrightarrow Z_G(\lambda) = 0. \quad \lightning$$

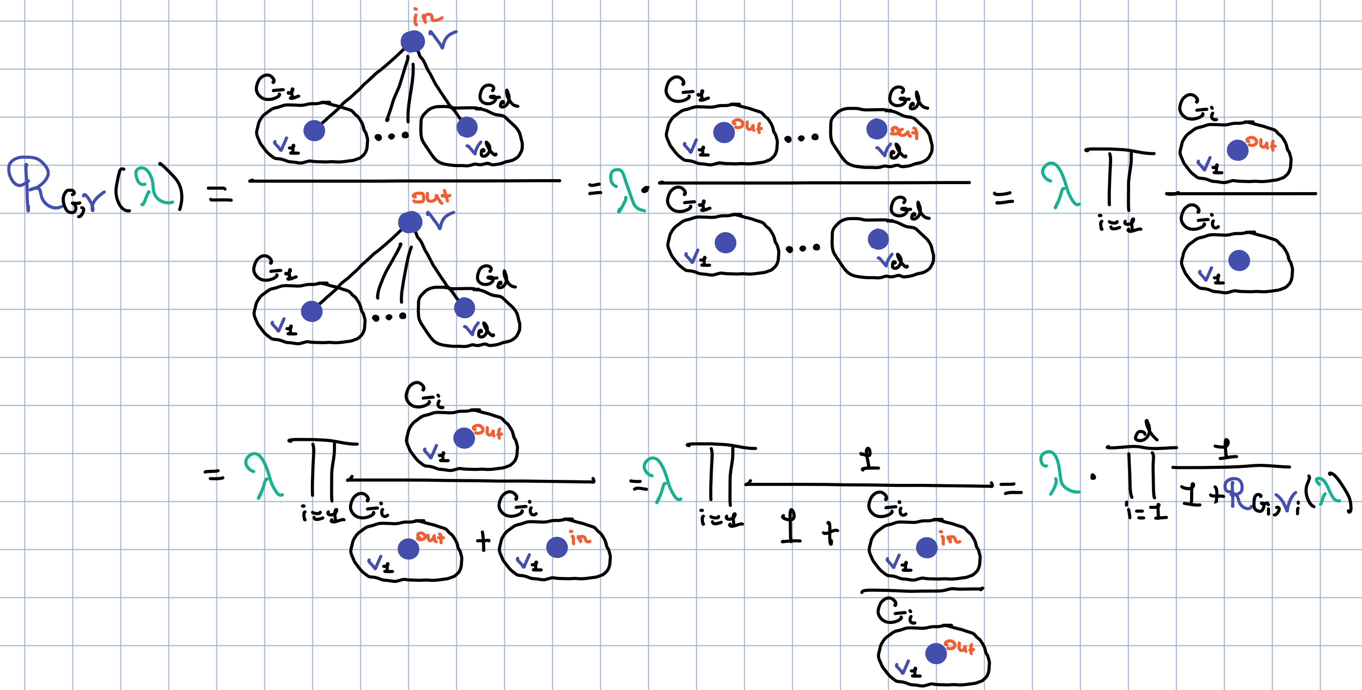
So R_Δ avoids three distinct values on U and is thus a normal family on U by Montel's Theorem. So $\lambda_0 \notin A_\Delta$.

A recursion for Ratios

Lemma: Let $(G_1, v_1), \ldots, (G_d, v_d)$ be a sequence of rooted graphs. Construct the rooted graph (G, v) by attaching a new vertex v to all roots v_i. Then:

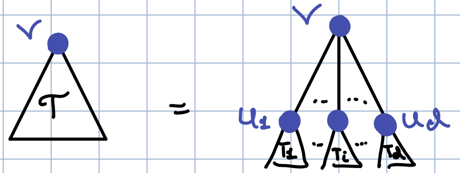
$$R_{G,v}(\lambda) = \lambda \cdot \prod_{i=1}^{d} \frac{1}{1 + R_{G_i,v_i}(\lambda)}.$$

Proof:

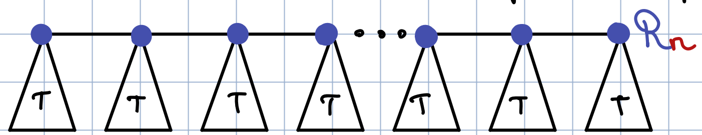


Implementation

Let (T, v) be a rooted tree, so $R_{T,v}(\lambda) = \lambda \prod_{i=1}^{d} \frac{1}{1+R_{T_i u_i}(\lambda)}$



Consider the graph P_n consisting of a path on n vertices with a copy of (T, v) implemented at each vertex. Let R_n denote the ratio at one of the endpoints.



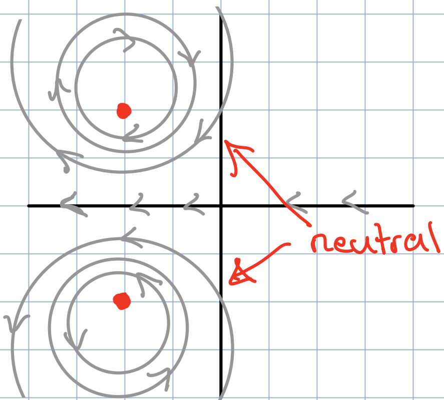
Then $R_n(\lambda) = \lambda \left[\prod_{i=1}^{d} \frac{1}{1+R_{T_i u_i}(\lambda)} \right] \cdot \frac{1}{1+R_{n-1}(\lambda)} = \frac{R_{Tv}(\lambda)}{1+R_{n-1}(\lambda)} = f_{R_{Tv}(\lambda)}\left(R_{n-1}(\lambda)\right) = f_{R_{Tv}(\lambda)}^{\circ n}(0)$, where

$f_{\mu}(z) = \frac{\mu}{1+z}$.

Dynamics of f_μ

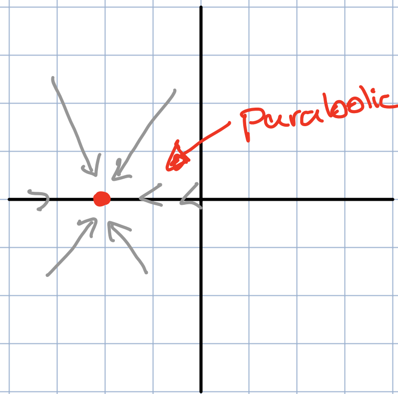
The dynamic behaviour of $f_\mu(z) = \frac{\mu}{1+z}$ for different values of μ:

Elliptic



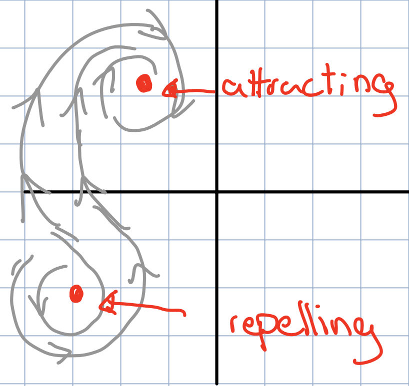
$\mu < -1/4$

Parabolic



$\mu = -1/4$

Loxodromic

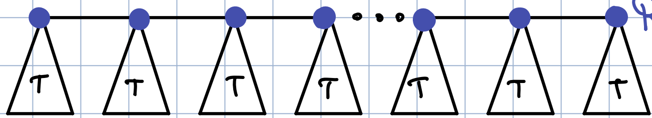


$\mu \in \mathbb{C} \setminus \mathbb{R}_{\leq -1/4}$

Theorem: The closure of the zero-locus equals the activity-locus, i.e. $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$.

Proof: We proved $\overline{\mathcal{Z}_\Delta} \supseteq \mathcal{A}_\Delta$. Let $\Delta \geq 3$, take some $\lambda_0 \in \mathcal{Z}_\Delta$ and take any neighborhood U of λ_0.

Then there exists a rooted tree (T, v) with $R_{T,v}(\lambda_0) = -1$ with max degree at most Δ and $\deg(v) = 1$. Consider the ratios of paths with (T, v) implemented.



$R_n(\lambda) = f^{\circ n}_{R_{T,v}(\lambda)}(0)$

Suppose (a subsequence of) $f^{\circ n}_{R_{T,v}(\lambda)}(0)$ converges to a holomorphic function $U \to \hat{\mathbb{C}}$.

For $\lambda \in U \cap R_{T,v}^{-1}(\mathbb{C} \setminus \mathbb{R}_{\leq -1/4})$ we see that $f^{\circ n}_{R_{T,v}(\lambda)}(0)$ converges to a fixed point of $f_{R_{T,v}(\lambda)}$, which, by analytic continuation, it must do for every $\lambda \in U$.

However $f^{\circ n}_{R_{T,v}(\lambda_0)}(0) = f^{\circ n}_{-1}(0)$ is real for all n and thus cannot converge to the non-real fixed points of f_{-1}. So $\mathcal{R}_\Delta$ is not normal on U. So $\lambda_0 \in \mathcal{A}_\Delta$.

Fast implementation

We say that there exists a fast implementation algorithm for $\lambda_0 \in \mathbb{C}$ if there is a polynomial time algorithm for:

#Input: A parameter $P \in \mathbb{C}$ and $\varepsilon > 0$.

#Output: A rooted tree $(T, v) \in \mathcal{G}_\Delta$ with $|R_{T,v}(\lambda_0) - P| < \varepsilon$.

Theorem [Bezakova, Galanis, Goldberg, Štefankovič 2020]:
If there exists a fast implementation algorithm for $\lambda_0 \in \mathbb{C}$ then approximating $Z_G(\lambda_0)$ for $G \in \mathcal{G}_\Delta$ is #P-hard.

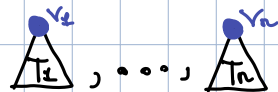
Theorem: The parameters for which there exists a fast implementation algorithm lie dense in Λ_Δ.

#Input: A parameter $\mathcal{P}\in\mathbb{C}$ and $\varepsilon>0$.

#Output: A rooted tree $(T,v)\in\mathcal{G}_\Delta$ with $|R_{T,v}(\lambda_0)-\mathcal{P}|<\varepsilon$.

Theorem: The parameters for which there exists a fast implementation algorithm lie dense in $\mathcal{A}_\Delta$.

Step 1: Parameters λ_0 for which $\{R_{G,v}(\lambda_0) : G\in\mathcal{G}_\Delta \text{ and } v\in V(G)\}$ is dense in $\mathbb{C}$ lie dense in $\mathcal{A}_\Delta$.

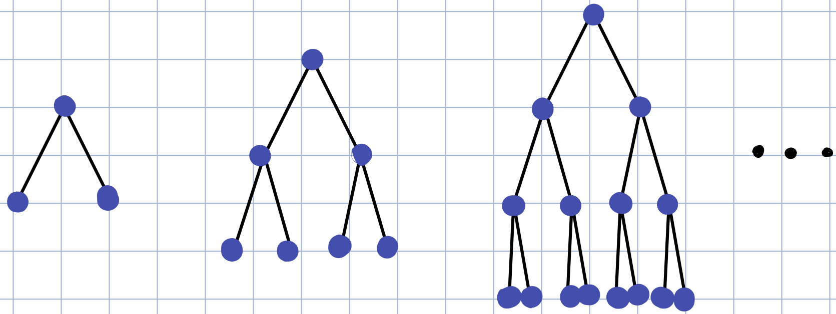
Step 2: Given Step 1 we can pick a finite set of rooted trees $\{$ 

$\}$ whose ratios $\{\mu_1,\dots,\mu_n\}$ correspond to a set of möbius transformations $\{f_{\mu_1},\dots,f_{\mu_n}\}$ for which we have an algorithm that, given $\mathcal{P}\in\mathbb{C}$, $\varepsilon>0$ yields a sequence of indices $i_1,\dots,i_m$ for which $|(f_{\mu_{i_1}}\circ\cdots\circ f_{\mu_{i_m}})(0)-\mathcal{P}|<\varepsilon$, where $m\sim \mathrm{poly}(\mathrm{Size}(\mathcal{P},\varepsilon))$.

Location of the zeros (Z_Δ)

An important family of graphs within $\mathcal{G}_\Delta$ are rooted $(\Delta-1)$-ary trees.

Example ($\Delta=3$):



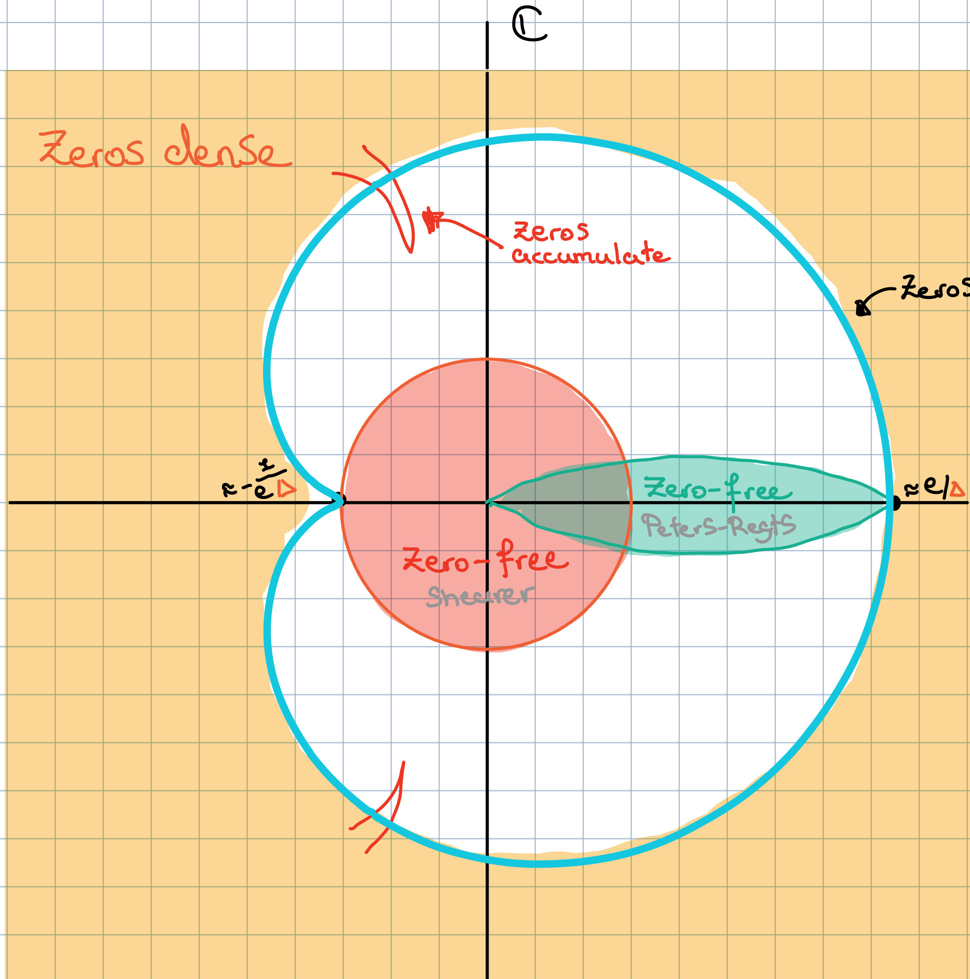
The ratio at the root of such a tree of depth n is given by $f^{\circ n}_{\Delta-1,\lambda}(0)$ where $f_{d,\lambda}(z) = \frac{\lambda}{(1+z)^d}$.

We define

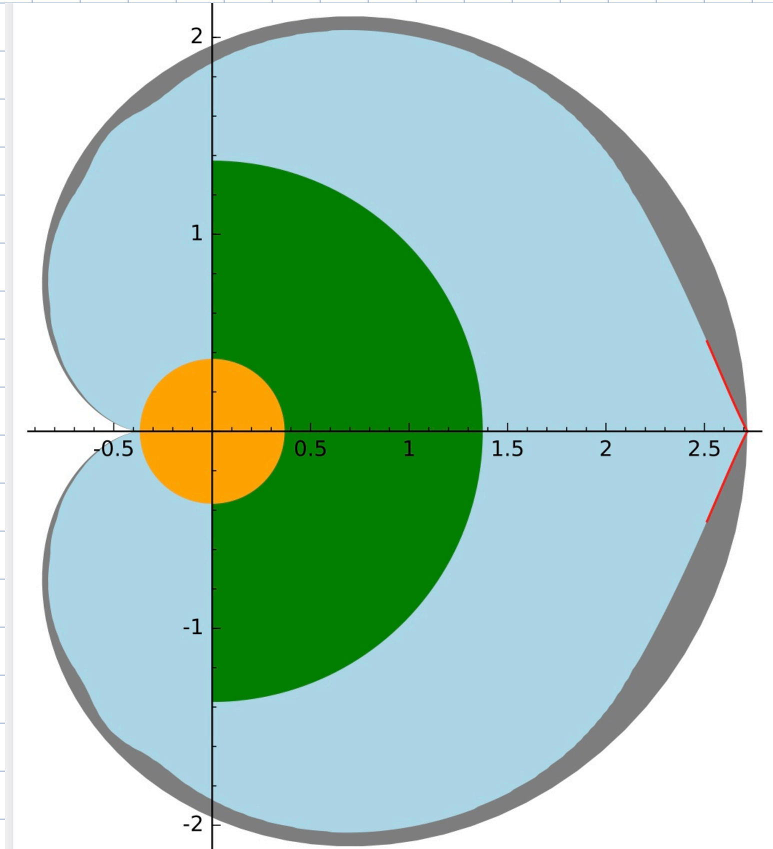
$$U_\Delta = \{\lambda \in \mathbb{C} : f_{\Delta-1,\lambda} \text{ has an attracting fixed point}\}.$$

This set is colloquially known as the cardioid.

Location of the zeros (Z_Δ)



Limit $\Delta \to \infty$



Limit $\Delta \to \infty$

Theorem [Weitz 2006]:

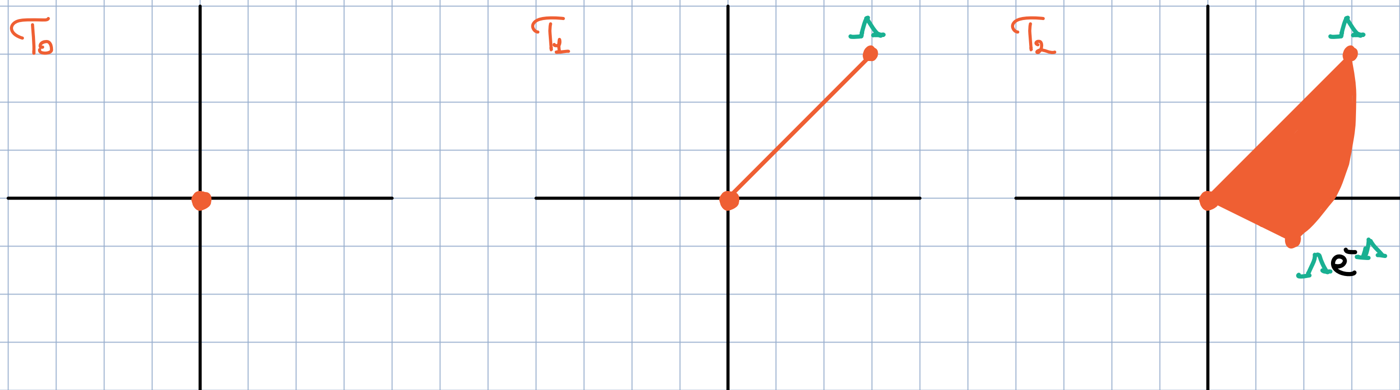
Given $\lambda \in \mathbb{C}$, $\Delta \in \mathbb{Z}_{\geq 2}$. Let $S_0 = \{0\}$ and $S_{n+1} = S_n \cup \{\lambda \prod_{i=1}^{\Delta-1} \frac{1}{1+z_i} : z_i \in S_n\}$. Then

$$\bigcup_{n=0}^{\infty} S_n \overset{*}{=} \{R_{G,v}(\lambda) : G \in \mathcal{G}_{\Delta} \text{ and } v \in V(G)\}.$$

We let $\Lambda = \Delta \cdot \lambda$ and $Z_i = \Delta \cdot z_i$, taking the limit we define the "analagous" sets.

Definition: Given $\Lambda \in \mathbb{C}$ we define $T_0 = \{0\}$ and $T_{n+1} = \mathrm{ConvHull}[T_n \cup \{\Lambda e^{-z} : z \in T_n\}]$.

Then let $V_\Lambda = \bigcup_{n=0}^{\infty} T_n$.



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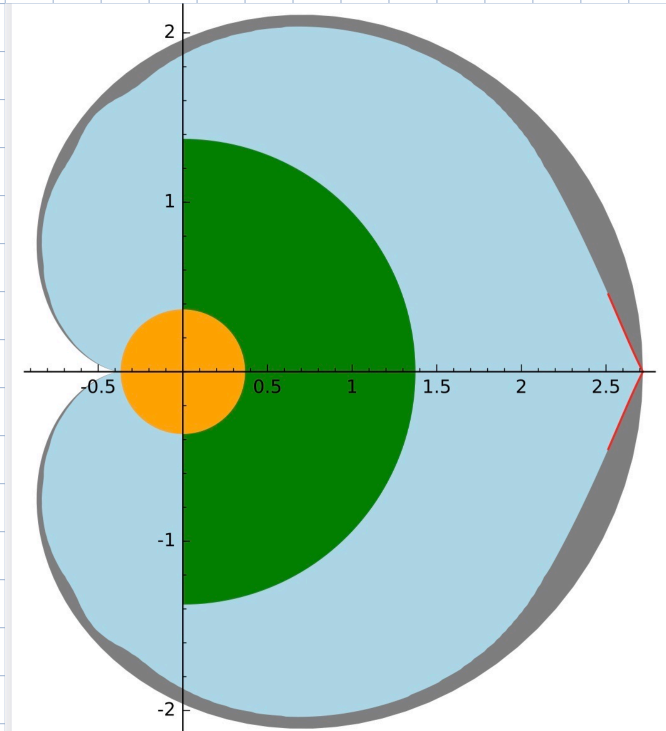
Definition: Let $V_\infty = \{\Lambda \in \mathbb{C} : V_\Lambda \text{ is bounded}\}$.

Theorem [Bencs, B., Peters]:

We have $\lim_{\Delta \to \infty} \Delta \cdot (Z_\Delta)^c \overset{*}{=} V_\infty$. Moreover V_∞ is star-convex from 0 and ∂V_∞ intersects the boundary of $\lim_{\Delta \to \infty} \Delta \cdot U_\Delta$ only in $\{-1/e, e\}$.

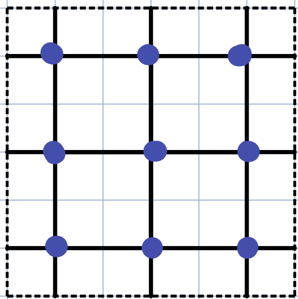
Limit $\Delta \to \infty$



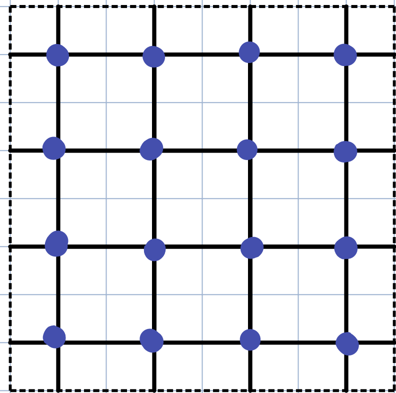
Q. Is the boundary of V_∞ piecewise analytic?

Zero-Locus vs Activity-Locus

For $n \geq 3$ let T_n be the $n \times n$ torus graph.

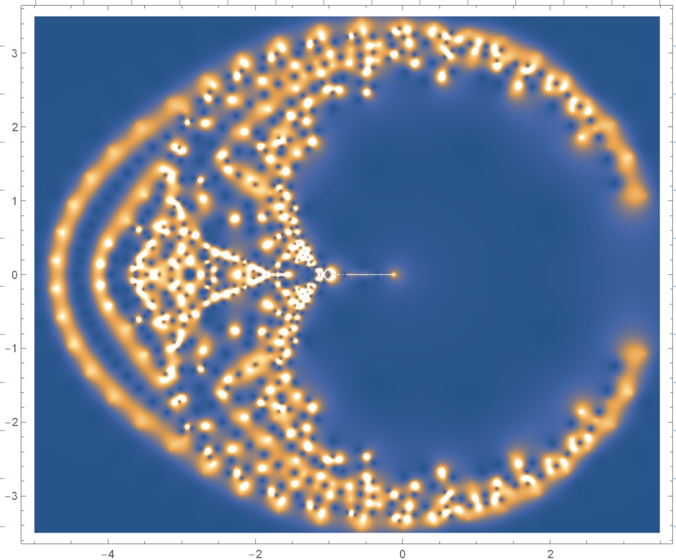


T_3



T_4

Q. Could it be true that the zero-locus of $\{T_n\}_{n\geq 3}$ and its activity locus are equal? If so, why?



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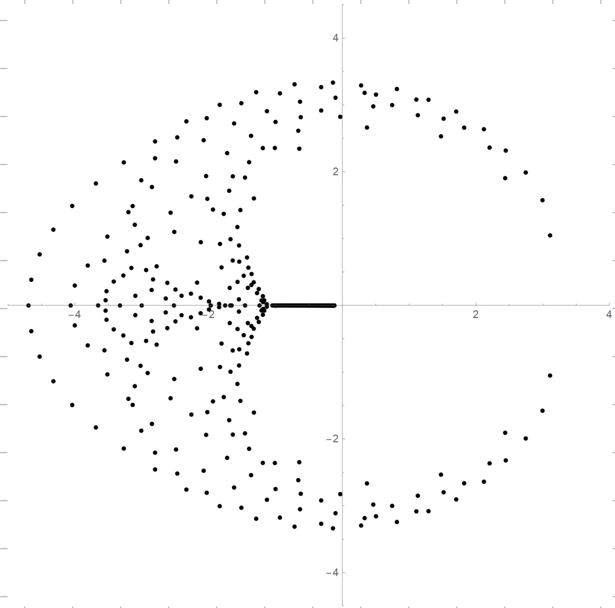

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