Lee-Yang zeros and the complexity of the ferromagnetic Ising Model on bounded-degree graphs

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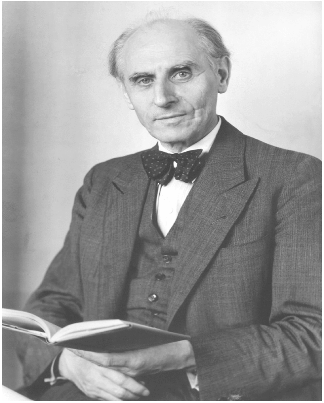
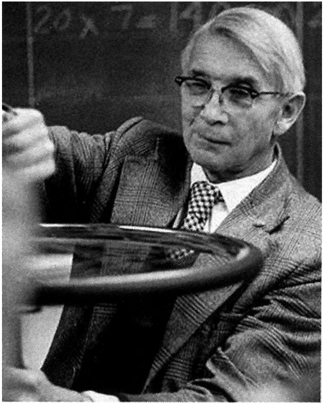
Very short summary

$$\frac{\text{Values for which the Ising partition function is zero.}}{} = \frac{}{\text{Values for which approximating the Ising partition function is \#P-hard.}}$$

The Ising Model



Wilhelm Lenz (1888 - 1957)



Ernst Ising (1900 - 1998)

The Ising Model

- For a graph $G = (V, E)$ a *configuration* is a map $\sigma : V \to \{1, -1\}$.
- The energy of a configuration is

$$\mathcal{E}(\sigma) = -H \sum_{v \in V} \sigma(v) - J \sum_{u \sim v \in E} \sigma(u) \cdot \sigma(v),$$

  where $J > 0$.
- Let $n_+(\sigma) = \sigma^{-1}(1)$ and $\delta(\sigma) = \{\{u, v\} \in E : \sigma(u) \neq \sigma(v)\}$, then

$$\sum_{v \in V} \sigma(v) = |n_+(\sigma)| - (|V| - |n_+(\sigma)|) = 2|n_+(\sigma)| - |V|$$

  and

$$-\sum_{u \sim v \in E} \sigma(u) \cdot \sigma(v) = 2|\delta(\sigma)| - |E|.$$

- Therefore

$$\mathcal{E}(\sigma) = -2H \cdot (|n_+(\sigma)| - |V|/2) + 2J \cdot (|\delta(\sigma)| - |E|/2)$$

The Ising Model

$$\mathcal{E}(\sigma) = -2H \cdot (|n_+(\sigma)| - |V|/2) + 2J \cdot (|\delta(\sigma)| - |E|/2)$$

- The probability of state σ is proportional to

$$W(\sigma) = e^{-\beta \cdot \mathcal{E}(\sigma)} = (e^{2H\beta})^{|n_+(\sigma)| - |V|/2} \cdot (e^{-2J\beta})^{|\delta(\sigma)| - |E|/2}.$$

- We let $\lambda = e^{2H\beta}$ and $b = e^{-2J\beta}$.
- The probability of state σ occuring is $W(\sigma)/\tilde{Z}$, where

$$\tilde{Z} = \sum_{\sigma : V \to \{+,-\}} W(\sigma) = \lambda^{-|V|/2} \cdot b^{-|E|/2} \underbrace{\sum_{\sigma : V \to \{+,-\}} \lambda^{|n_+(\sigma)|} b^{|\delta(\sigma)|}}_{Z_G(\lambda, b)}$$

Approximating $Z_G(\lambda, b)$

Let $\mathbb{C}_{\mathbb{Q}} = \{z \in \mathbb{C} : \mathrm{re}(z), \mathrm{im}(z) \in \mathbb{Q}\}$.
Let $\lambda \in \mathbb{C}_{\mathbb{Q}}$, $b \in (0,1) \cap \mathbb{Q}$, $K \in \mathbb{Q}_{\geq 1}$ and $\Delta \in \mathbb{Z}_{\geq 2}$.
We consider the following problems.

Name $\#\mathrm{IsingNorm}(\lambda, b, \Delta, K)$.
Instance A graph $G = (V, E)$ with maximum degree $\leq \Delta$.
Output If $Z_G(\lambda, b) = 0$, the algorithm may output any rational. Otherwise, it must return a rational $\widehat{N}$ such that $\widehat{N}/K \leq |Z_G(\lambda, b)| \leq K\widehat{N}$.

and for $\rho \in \mathbb{Q}_{\geq 0}$

Name $\#\mathrm{IsingArg}(\lambda, b, \Delta, \rho)$.
Instance A graph $G = (V, E)$ with maximum degree $\leq \Delta$.
Output If $Z_G(\lambda, b) = 0$, the algorithm may output any rational. Otherwise, it must return a rational $\widehat{A}$ such that $|\widehat{A} - a| \leq \rho$ for some $a \in \arg(Z_G(\lambda, b))$.

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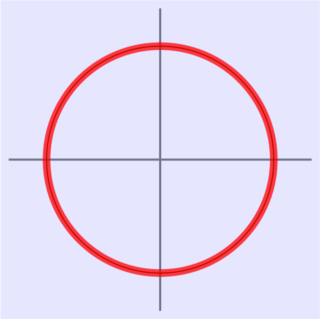
A *fully polynomial time approximation scheme* (FPTAS) for approximating $Z_G(\lambda, b)$ is an algorithm that for any n-vertex graph G of maximum degree at most Δ and any rational $\varepsilon > 0$ solves both problems $\#\text{IsingNorm}(\lambda, b, \Delta, 1 + \varepsilon)$ and $\#\text{IsingArg}(\lambda, b, \Delta, \varepsilon)$ in time polynomial in n/ε.

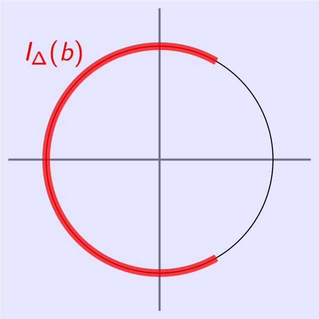
Approximation schemes and Lee-Yang zeros

The Lee-Yang theorem (1952) states that for fixed $b \in (0,1)$ the complex zeros of $Z_G(\lambda, b)$ for any graph G lie on the unit circle.
Peters and Regts showed in 2018 that the situation for graphs with with maximum degree $\leq \Delta$ is as follows:

$0 < b \leq 1 - 2/\Delta$



$1 - 2/\Delta < b < 1$



Zeros are dense and contained in the red arcs

Approximation schemes and Lee-Yang zeros

- Liu, Sinclair, and Srivastava (2018) obtained an FPTAS for approximating $Z_G(\lambda, b)$ for $\lambda \notin \mathbb{S}$.
- Using the methods by Barvinok (2017) and Patel and Regts (2017) Peters and Regts obtained an FPTAS for approximating $Z_G(\lambda, b)$ for $1 - 2/\Delta < b < 1$ and $\lambda \in \mathbb{S} \setminus I_\Delta(b)$.

Our main result is the following:

Theorem (B., Galanis, Patel, Regts)

Let $\Delta \geq 3$ be an integer and let $K = 1.001$ and $\rho = \pi/40$.

(a) *Let $b \in \left(0, \frac{\Delta-2}{\Delta}\right]$ be a rational, and $\lambda \in \mathbb{C}_{\mathbb{Q}} \cap \mathbb{S}$ such that $\lambda \neq \pm 1$. Then the problems $\#\text{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\text{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard.*

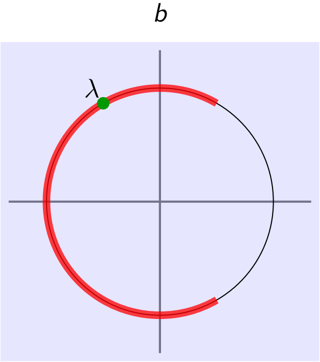
(b) *Let $b \in \left(\frac{\Delta-2}{\Delta}, 1\right)$ be a rational. Then the collection of complex numbers $\lambda \in \mathbb{C}_{\mathbb{Q}} \cap I_\Delta(b)$ for which $\#\text{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\text{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard is dense in the arc $I_\Delta(b)$.*

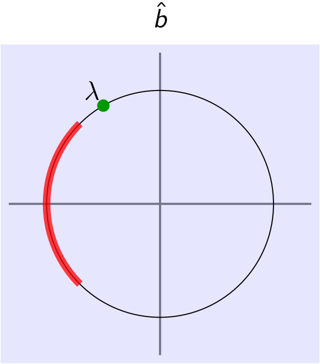
#P-Hardness

- #P is a complexity class of counting problems.
  - What is the value of the permanent of a given matrix consisting of 1s and 0s?
  - How many perfect matchings are there in a given bipartite graph?
- #IsingNorm(λ, b, Δ, K) being #P-hard implies that if there is a polynomial time algorithm to solve #IsingNorm(λ, b, Δ, K), then any problem in #P can be solved in polynomial time.
- We show that a polynomial time algorithm for #IsingNorm(λ, b, Δ, K) can be used to solve the problem of calculating $Z_G(\lambda, \hat{b})$ given a 3-regular graph G in polynomial time.
- This problem is known to be #P-hard [Kowalczyk-Cai '11].

Very rough idea of the reduction

This $\hat{b}$ is chosen to have the property that $Z_G(\lambda, \hat{b})$ cannot be zero.

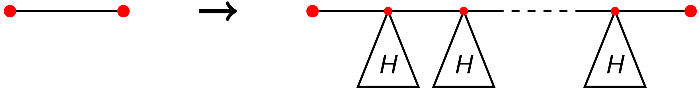




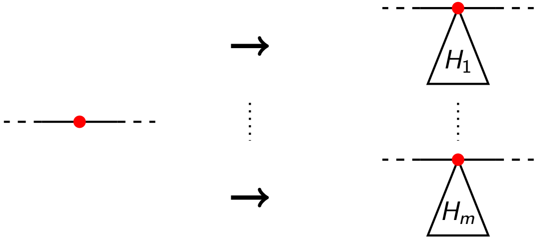
Very rough idea of the reduction

We transform the input graph G in multiple ways. Involving steps like:

- We replace edges of G by paths with gadgets to simulate edge activity $\hat{b}$.

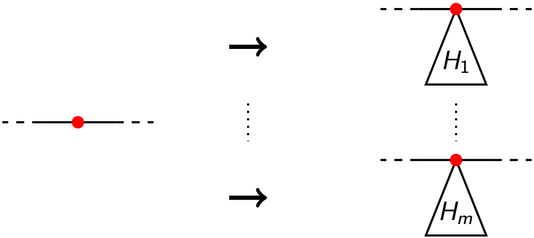


- We *probe* degree 2 vertices with multiple gadgets.



A polynomial amount of applications of $\#\text{IsingNorm}(\lambda, b, \Delta, K)$ to these transformed graphs allow us to calculate $Z_G(\lambda, \hat{b})$ exactly.

Very rough idea of the reduction



- We need our gadgets to exist within the family of rooted graphs with bounded degree Δ and root degree 1.
- We need our gadgets to be *small* compared to the size of the input graph G.

Ratios/fields

- Recall that

$$Z_G(\lambda) = \sum_{\sigma:V\to\{+,-\}} \lambda^{|n_+(\sigma)|} b^{|\delta(\sigma)|}.$$

- For a graph G and a vertex $v \in V$ we define

$$Z_{G,v^+}(\lambda) := \sum_{\sigma:V\to\{+,-\};\sigma(v)=+} \lambda^{|n_+(\sigma)|} b^{|\delta(\sigma)|}$$

  and we define $Z_{G,v^-}(\lambda)$ analogously.
- We then define the ratio

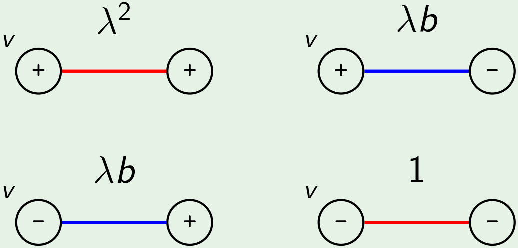
$$R_{G,v}(\lambda) = \frac{Z_{G,v^+}(\lambda)}{Z_{G,v^-}(\lambda)}.$$

Ratios/fields: Example

$$R_{G,v}(\lambda) = \frac{Z_{G,v^+}(\lambda)}{Z_{G,v^-}(\lambda)}$$

Example

Let G be an edge and v one of its endpoints.



So

$$R_{G,v}(\lambda) = \frac{\lambda^2 + \lambda b}{\lambda b + 1}.$$

Ratios/fields

These ratios are rational maps $\mathbb{S} \to \mathbb{S}$. Given a particular λ, for our reduction to work we need the following.

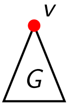
- We need the ratios to be dense in the unit circle, i.e. we want

$$\{R_{G,v}(\lambda) : G \text{ bounded degree } \Delta \text{ with } \deg(v) = 1\}$$

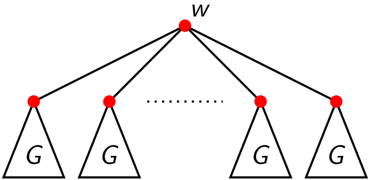
  to be dense in $\mathbb{S}$.
- We need exponentially fast implementation, i.e. we need an algorithm that, given a $P \in \mathbb{S}$ and $\epsilon > 0$, yields a rooted graph (G, v) such that
  - G has its degree bounded by Δ and $\deg(v) = 1$;
  - $|R_{G,v}(\lambda) - P| < \epsilon$;
  - the size of G is $\mathcal{O}(\log(1/\epsilon))$.

Graph constructions

Suppose we have a rooted graph (G, v) with ratio $R_{G,v}(\lambda)$.



We construct a new graph $\tilde{G}$ by attaching k disjoint copies of G to a new root w.



We have $R_{\tilde{G},w} = f_{k,\lambda}(R_{G,v}(\lambda))$, where

$$f_{k,\lambda}(z) = \frac{\lambda}{(1+z)^k}.$$

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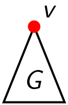

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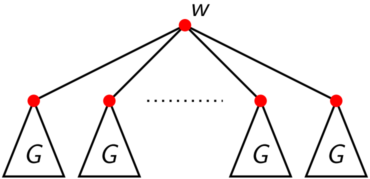


We have $R_{\tilde{G},w}(\lambda) = f_{k,\lambda}(R_{G,v}(\lambda))$, where

$$f_{k,\lambda}(z) = \lambda \left(\frac{z+b}{bz+1} \right)^k .$$

Graph constructions

Let $f_\lambda(z) = f_{\lambda,1}(z) = \lambda\left(\frac{z+b}{bz+1}\right)$. Then the ratio of a path on n vertices is

$$f_\lambda^{\circ n}(1)$$

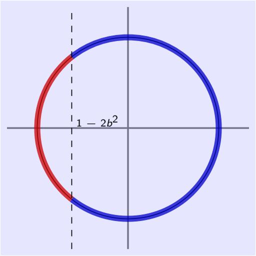
If (G, v) is a rooted graph with ratio $\mu = R_{G,v}(\lambda)$ then the ratio is

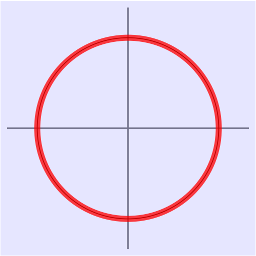
$$f_\mu^{\circ n}(1)$$

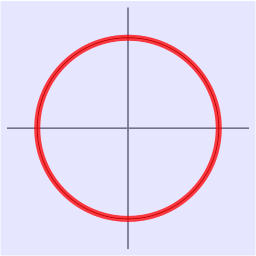
If $(G_1, v_1)\ldots(G_n, v_n)$ are rooted graphs with ratios $\mu_1, \ldots, \mu_n$ then

$$\left(f_{\mu_n} \circ \cdots \circ f_{\mu_1}\right)(1)$$

The Möbius transformation $f_\mu(z) = \mu\left(\frac{z+b}{bz+1}\right)$







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